QUALIFYING EXAMINATION IN REAL ANALYSIS
August 13, 2010
1:30–3:30 pm

The six problems are weighted equally. $A^c$ denotes the complement of the set $A$, and $m$ denotes Lebesgue measure.

1. For small values of $|x|$, which is larger, $\sin^2 x$ or $\sin(x^2)$?

2. Let $f \in L^1([0, 1])$, and for $x \in [0, 1]$ define
   
   $$g(x) = \int_x^1 \frac{f(t)}{t} \, dt.$$ 

   Show that $g \in L^1([0, 1])$ and that $\int_0^1 g(x) \, dx = \int_0^1 f(x) \, dx$.

3. Let $A \subset [0, 1]$ be measurable, and define $L^2(A) \subset L^2([0, 1])$ to be the subspace consisting of all $f \in L^2$ such that $f \equiv 0$ a.e. off $A$. Show that
   $$L^2(A)^\perp = L^2(A^c).$$

4. Let $f_n \to f$ pointwise a.e. on $[0, 1]$. Suppose
   $$\limsup_{n \to \infty} \|f_n\|_1 \leq \|f\|_1 < \infty.$$ 

   Show that $f_n \to f$ in $L^1$, i.e., that $\lim_{n \to \infty} \|f_n - f\|_1 = 0$.

5. Suppose $f \in L^1(\mathbb{R})$ and $f \geq 0$. For $y > 0$, let
   $$g(y) = m(\{x \in \mathbb{R} : f(x) \geq y\}).$$ 

   Set $G(y) = yg(y)$.
   a. Prove that $\lim_{y \to 0^+} G(y) = \lim_{y \to \infty} G(y) = 0$ and that $G$ is bounded.
   b. Prove that $G$ achieves its maximum at some point $y_0$.

6. Let $f_1, f_2, \ldots \colon \mathbb{N} \to \mathbb{R}$ be a sequence of functions such that $|f_i(n)| \leq n$ for all $i, n \in \mathbb{N}$. Show that there is a subsequence $f_{i'}$ converging pointwise on $\mathbb{N}$ to a function $f_0$. 