Real Analysis Qualifying Examination
August 2008

There are five problems, each worth 20 points. Give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

In Questions 2 and 3 below m stands for Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^2$ respectively.

1. Let $\{c_n\}$ be a sequence of complex numbers such that $\sum_{n=-\infty}^{\infty} |c_n| < \infty$. Define a function f by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}.$$

   (a) Prove that f is a continuous function on $\mathbb{R}$, that $f(x+1) = f(x)$ for all x, and that

$$\int_0^1 f(x) e^{-2\pi i m x}\, dx = c_m.$$

   (b) Prove that if, for some irrational number α, we further have that $f(x+\alpha) = f(x)$ for all x, then f must be a constant function.

2. (a) Let I be an interval of real numbers. If $E \subseteq I$ is measurable with $m(E) \geq \frac{5}{6} m(I)$, show that

$$m\left(E \cap (E-d)\right) > \tfrac{1}{2} m(I)$$

   for all $|d| < \frac{1}{6} m(I)$.

   (b) Now suppose that $E \subseteq \mathbb{R}$ is measurable with $m(E) > 0$. Prove that

$$m\left(E \cap (E-d) \cap (E-2d)\right) > 0$$

   for all d in a sufficiently small interval centered at the origin.

3. (a) Show that there exists an absolute constant $C_1 > 0$ so that

$$m\left(\{(x,y) \in \mathbb{R}^2 : x^4 + y^2 < t\}\right) = C_1\, t^{1/4+1/2}$$

   for all $t > 0$.

   (b) Using part (a), or otherwise, find all the values of $p > 0$ for which

$$\int_{-1}^{1} \int_{-1}^{1} \left(\frac{1}{x^4+y^2}\right)^p dx\, dy < \infty.$$

4. Let $E \subseteq \mathbb{R}^n$ be a measurable set, and $\{f_n\}$ be a sequence of measurable functions on E for which

$$\lim_{n\to\infty} \int_E |f_n(x)|\, dx = A$$

and

$$\lim_{n\to\infty} f_n(x) = g(x)$$

for almost every $x \in E$, where

$$\int_E |g(x)|\, dx = B.$$

(a) Prove that

$$\lim_{n\to\infty} \int_E |f_n(x) - g(x)|\, dx = A - B.$$

Hint: Use the fact that

$$|f_n(x)| - |g(x)| \leq | f_n(x) - g(x)| \leq |f_n(x)| + |g(x)|.$$

(b) Give an example of a sequence $\{f_n\}$ of such functions for which $A \neq B$.

5. Let

$$\mathcal{C} = \left\{ f \in L^2(0,1) \,:\, \int_0^1 f(x)\, dx = 1 \text{ and } \int_0^1 x f(x) = 2 \right\}$$

(a) Let $g(x) = 18x^2 - 5$. Show that $g \in \mathcal{C}$ and that

$$\mathcal{C} = g + \mathcal{S}^\perp$$

where $\mathcal{S}^\perp$ denotes the orthogonal complement of $\mathcal{S} = \mathrm{Span}\,(\{1, x\})$.

(b) Find *the* function $f_0 \in \mathcal{C}$ for which

$$\int_0^1 |f_0(x)|^2 dx = \inf_{f\in\mathcal{C}} \int_0^1 |f(x)|^2 dx.$$