Real Analysis Qualifying Exam, Fall 2006
(2 hours)

Instructions: Work problems 1-3 and any 3 of the other 4 problems.

#1. (15 points) Let $B$ be a bounded subset of $\mathbb{R}^n$. Prove the following:
a) If $B$ is closed in $\mathbb{R}^n$ and $f : B \to \mathbb{R}$ is a continuous function, then $f$ is uniformly continuous.
b) If $B$ is not closed in $\mathbb{R}^n$, then there exists a continuous $f : B \to \mathbb{R}$ which is not uniformly continuous.

#2. (10 points) Show that the sequence of functions $f_n = n^{-1} \chi_{(0,n)}$ converges uniformly to the 0-function on $\mathbb{R}$ but does not converge to 0 in $L^1$. Here $n = 1, 2, \ldots$ and $\chi_{(0,n)}$ denotes the characteristic function of $(0, n) \subset \mathbb{R}$.

#3. (15 points) Let $(f_k)_{k=1}^{\infty}$ be a uniformly bounded sequence of functions on $[0, 1]$ which converges pointwise to a function $f$. Answer the following with a brief explanation or a counterexample:
a) If each $f_k$ is Lebesgue integrable, does it follow that $f$ is Lebesgue integrable?
b) If each $f_k$ is Riemann integrable, does it follow that $f$ is Riemann integrable?

#4. (20 points) Let $m$ be Lebesgue measure on $\mathbb{R}^d$, and let $(f_k)_{k=1}^{\infty}$ be a sequence of measurable functions defined on a measurable set $E$ with $m(E) < \infty$ such that $f_k \to f$ pointwise on $E$. Prove that for any $\varepsilon > 0$ there exists a subset $A$ of $E$ such that $m(E - A) \leq \varepsilon$ and $f_k \to f$ uniformly on $A$.

#5. (20 points) Define the norm on $L^2(\mathbb{R})$ and give an argument for the completeness of $L^2(\mathbb{R})$.

#6. (20 points)
a) Show that if $f \in L^1(\mathbb{R}^d)$, if $\hat{f}$ is defined on $\mathbb{R}^d$ by $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$, then $\hat{f}$ is a continuous function.
b) Indicate how to obtain an example of $f \in L^1(\mathbb{R})$, such that $\hat{f}$ is not a differentiable function. [No justification required.]

#7. (20 points) If $m$ is Lebesgue measure on $\mathbb{R}$ and $f \in L^1(\mathbb{R})$, let $F(x) = \int_{(-\infty, x]} f dm$, for $x \in \mathbb{R}$. Show that for almost every $x$, $F'(x)$ exists and is equal to $f(x)$. [Give the main points of a proof and indicate some of the details.]