Real Analysis Qualifying Exam – Spring 2026

All problems are of equal weight (5 points). Please arrange your solutions in numerical order even if you do not solve them in that order. Show work and carefully justify/prove your assertions.

1. For $n \in \mathbb{Z}_+$ and $x \in \mathbb{R}$, define
$$f_n(x) = \frac{x}{1+nx^2}.$$
   (a) Show that $(f_n)_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$ to a function f, and determine f.
   (b) Is it true that $f_n'(x) \to f'(x)$ for almost every $x \in \mathbb{R}$?
   (c) Let $X = \{x \in \mathbb{R} : \lim_{n\to\infty} f_n'(x) = f'(x)\}$. Is it true that $f_n'(x) \to f'(x)$ uniformly on X?

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and let f be a measurable function. Prove that $f \in L^1(X)$ if and only if
$$\sum_{k\in\mathbb{Z}} 2^k \mu\big(\{x \in X : 2^k \le |f(x)| < 2^{k+1}\}\big) < \infty.$$

3. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Then, for $f \in L^2(X)$ we have
$$\|f\|_{L^2(X)}^2 = 2\int_0^{\infty} \alpha\mu(\{x \in X : |f(x)| > \alpha\})\, d\alpha.$$
[Hint: Fubini's theorem may be useful.]

4. Let $\mathcal{H}$ be a Hilbert space.
   (a) Let $M_1 \le \mathcal{H}$ be a finite dimensional subspace and let $S\colon M_1 \to \mathcal{H}$ be a linear operator. Prove that S is a bounded operator, that is there is a constant $C > 0$ such that $\|Sf\| \le C\|f\|$ for all $f \in M_1$.
   (b) Let $f_1, \dots, f_n \in \mathcal{H}$ and $g_1, \dots, g_n \in \mathcal{H}$ such that $f_1, \dots, f_n$ are linearly independent. Prove that there exists a bounded linear operator $T\colon \mathcal{H} \to \mathcal{H}$ such that $Tf_k = g_k$ for all $1 \le k \le n$.
   [Hint for part (b): You can use the orthogonal projection to the subspace spanned by the vectors $f_1, \dots, f_n$ in constructing the operator T, or equivalently, you can use the orthogonal decomposition of $\mathcal{H}$ into this subspace and its orthogonal complement.]

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $(f_n)_{n=1}^{\infty}$, $(g_n)_{n=1}^{\infty}$ be two sequences of measurable functions, and assume that $f_n \to f$ and $g_n \to g$ in measure, for some measurable functions f and g.
   (a) If f, g are bounded show that $f_n g_n \to fg$ in measure.
   (b) If $\mu(X) < \infty$, then $f_n g_n \to fg$ in measure.
   [Hint for part (b): reduce to part (a).]

6. Let $\mathbb{T}$ be a torus and let $f \in L^2(\mathbb{T})$, $k \in L^1(\mathbb{T})$.
   (a) Show that
$$Tf(x) = \int_{\mathbb{T}} f(x-y)k(y)\, dy,$$
   converges for almost every $x \in \mathbb{T}$.
   (b) Show that $\|Tf\|_2 \le \|k\|_1 \|f\|_2$.
   (c) Compute the Fourier series of Tf, that is $\widehat{Tf}(m)$ for $m \in \mathbb{Z}$, where
$$\widehat{Tf}(m) = \int_{\mathbb{T}} Tf(x) e^{-2\pi i x m}\, dx.$$
   (d) Show that $\|\hat{k}\|_\infty \le \|k\|_1$ and
$$\sup_{\|f\|_2 \le 1} \|Tf\|_2 = \|\hat{k}\|_\infty.$$