Real Analysis Qualifying Exam

August, 2025

State clearly which theorem you are using. Conclusions in one problem in this exam may be used for another without proof. Each problem is worth 10 points.

Notation: m is the Lebesgue measure on the space $\mathbb{R}^n$, and so is dx.

1. Let f be an $\mathbb{R}$-valued measurable function on $[0,1]$. Put $A := f^{-1}(\mathbb{Z})$, and for $n \in \mathbb{N}$, $f_n(x) := [\cos(\pi f(x))]^{2n}$. Show that A is measurable and $\lim_{n\to\infty} \int_0^1 f_n(x)dx == m(A)$.

2. For $x \neq 0$, define $f(x)$ by the series
$$f(x) = \sum_{n=0}^{\infty} e^{-n|x|}$$
(i) Let $d > 0$. Show that for $x \in (-\infty, -d) \cup (d, \infty)$, the series converges uniformly and $f(x)$ is uniformly continuous.
(ii) Show that for $x \in (-\infty, 0) \cup (0, \infty)$, the series is not uniformly convergent and $f(x)$ is not uniformly continuous.

3. (10 points) Let f be a bounded real valued function on $[a,b]$, Assume that
$$\sup\{\int_{[a,b]} \phi\, dm : \phi \text{ is simple, } \phi \leq f\} = \inf\{\int_{[a,b]} \psi\, dm : \psi \text{ is simple, } f \leq \psi\}.$$
Show that f is Lebesgue measurable.

4. Let f be Lebesgue integrable on $(0,a)$ and $g(x) = \int_x^a \frac{f(t)}{t} dt$. Show that g is measurable and integrable on $(0,a)$ and $\int_0^a g(x)dx = \int_0^a f(x)dx$.

5. Let $f \in L^1(\mathbb{R})$. Define
$$F(x,r) := \frac{1}{r} \int_{[x-r,x+r]} f(y)dy, \ (x,r) \in \mathbb{R} \times (0,\infty).$$
Show that F is continuous in (x,r).

6. (i) For $f \in L^1(\mathbb{R})$, $t \in \mathbb{R}$, define $\tau_t(f)(x) = f(x-t)$. Show that $t \mapsto \tau_t(f)$ is a continuous map from $\mathbb{R}$ to $L^1(\mathbb{R})$.
(ii) Let $f \in L^1(\mathbb{R})$ and g a bounded measurable function. Show that $h = f * g$ is uniformly continuous, where as usual, $h(y) := (f * g)(y) := \int_{\mathbb{R}} f(y-x)g(x)dx$.