1. (a) Show that the mean $\mu$ of a random variable $X$ has the property
$$\min_{c} E(X - c)^2 = E(X - \mu)^2 = V(X).$$

(b) Prove that for any r.v. $X$
$$E|X| = \int_{0}^{\infty} P(|X| \geq t)dt.$$

2. Suppose that $X$ and $Y$ are independent random variables with the same exponential density
$$f(x) = \theta e^{-\theta x}, \quad x > 0.$$ 
Show that the sum $X + Y$ and the ratio $X/Y$ are independent.

3. Given a square integrable r.v. $X$, show that for $\lambda \geq 0$,
$$P(X - EX \geq \lambda) \leq \frac{\sigma^2(X)}{\sigma^2(X) + \lambda^2}.$$ 

4. (a) Given a random variable $X$ with finite mean square. Let $\mathcal{D}$ be a $\sigma$-algebra. Show that $E[X|\mathcal{D}]$ is the minimizer of $E(X - \xi)^2$ over all $\mathcal{D}$-measurable r.v.s $\xi$, i.e.,
$$E(X - E[X|\mathcal{D}])^2 \leq E(X - \xi)^2$$
for all $\mathcal{D}$-measurable r.v.s $\xi$.

(b) Let $(\Omega, \mathcal{F}, P)$ denote a probability space. Suppose $f : \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}$ is a bounded $\mathcal{B}(\mathbb{R}^n) \times \mathcal{C}$ measurable function and $X$ be a $n$-dimensional $\mathcal{D}$ measurable random variable. Assume $\mathcal{C}$ and $\mathcal{D}$ are independent. If $g(x) := Ef(x, \omega)$, then
$$g(X) = E[f(X, \omega)|\mathcal{D}], \quad \text{a.s.}$$

5. Let $\{X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with $E|X_1| < \infty$. Show that
$$\lim_{n \rightarrow \infty} \frac{1}{n} E(\max_{1 \leq k \leq n} |X_k|) = 0.$$ 

6. Let $\{X_n\}$ be iid r.v.s. Then,
(a) $n^{-1} \max_{1 \leq i \leq n} |X_i| \rightarrow 0$ in probability if and only if $nP(|X_1| > n) = o(1)$.
(b) $n^{-1} \max_{1 \leq i \leq n} |X_i| \rightarrow 0$ a.s. if and only if $E|X_1| < \infty$.

7. Let $X_1, X_2, \ldots$ be a sequence of independent r.v.s with $EX_i = 0$. Let $S_n = X_1 + X_2 + \cdots + X_n$ and $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$. Show that $\phi(S_n)$ is an $\mathcal{F}_n$-submartingale for any convex $\phi$ provided that $E|\phi(S_n)| < \infty$ for all $n$. 