Probability Theory, Ph.D. Qualifying Exam, Fall 2025

1. For any random variable X with finite mean μ, prove that:

$$\min_c E[(X-c)^2] = E[(X-\mu)^2] = \mathrm{Var}(X).$$

2. Let X and Y be independent random variables with the same exponential density $f(x) = e^{-x}$ for $x > 0$. Show that the sum $X + Y$ and the ratio X/Y are independent.

3. Let $\{X_n\}$ be a sequence of i.i.d. random variables such that $P(|X_1| > c) > 0$ for all $c > 0$. Show that:

$$P\left(\limsup_{n\to\infty} |X_n| = \infty\right) = 1.$$

4. Let $\{X_n\}$ be i.i.d. random variables with mean μ and finite variance σ^2. Let $S_n = X_1 + \cdots + X_n$. Show that:

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \longrightarrow N(0,1) \quad \text{in distribution as } n \to \infty.$$

5. Let $\{X_n\}$ be a sequence of i.i.d. random variables with $E|X_1| < \infty$. Prove that:

$$\sum_{n=1}^{\infty} (-1)^n \frac{X_n}{n} \quad \text{converges almost surely.}$$

6. Let $\{X_n\}$ be a sequence of i.i.d. random variables with $E[X_1] = 0$ and $E|X_1|^\alpha < \infty$ for some $\alpha > 0$. Prove that for any $\beta > \alpha$:

$$\frac{1}{n^{1/\beta}} \max_{1\le k\le n} |X_k| \longrightarrow 0 \quad \text{in probability as } n \to \infty.$$