Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 22, 2016

WITH SOLUTIONS

Problem 1. Let a represent a digit from 1 to 9. Which a gives

$$\frac{a!}{aa + a^2} = 2016?$$

Here aa indicates concatenation of the digit a. For example, if $a = 1$, then $aa = 11$.

(A) 5 (B) 6 (C) 7 (D) 8 (E)$^\heartsuit$ 9

Solution. Notice that

$$\frac{a!}{aa + a^2} = \frac{(a-1)!}{11 + a} < \frac{(a-1)!}{10}$$

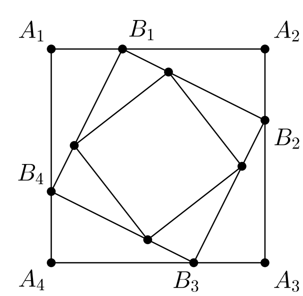
Remembering that $7! = 5040$, size estimation eliminates all choices but $a = 9$.

Problem 2. Suppose p and q are positive integers and $p < q$. How many of the following must be true? i) $p \leq q$ ii) $p < q + 1$ iii) $p \leq q + 1$ iv) $p + 1 < q$ v) $p + 1 \leq q$

(A) one (B) two (C) three (D)$^\heartsuit$ four (E) five

Solution. Four of them are true. Only (iv) $p+1 < q$ need not be true. For example, $2 < 3$, but $2 + 1$ is not less than 3. Notice that (v) $p + 1 \leq q$ must be true since p and q are integers.

Problem 3. Start with a square with vertices A_1, A_2, A_3, A_4. Let B_1 be the point 1/3 of the way from A_1 to A_2, and similarly for B_2, B_3, and B_4 (see figure). Repeat this construction on the quadrilateral $B_1B_2B_3B_4$ to construct another quadrilateral $C_1C_2C_3C_4$. What is the ratio of the area of $A_1A_2A_3A_4$ to the area of $C_1C_2C_3C_4$?



(A) 4 (B) $\frac{9}{5}$ (C)$^\heartsuit$ $\frac{81}{25}$ (D) $\frac{9}{4}$ (E) 9

Solution. Suppose the original square has side length 3, so that, for example, $A_1B_1 = 1$ and $B_1A_2 = 2$. Then $A_1A_2A_3A_4$ has area 9 and $B_1B_2B_3B_4$ has area 5 (this relies on the Pythagorean theorem and the fact that $B_1B_2B_3B_4$ is a square). So the ratio of the first area to the third area is $(9/5)^2 = 81/25$.

Problem 4. What is the 100th digit in the decimal expansion of $\frac{1}{7}$?

(A) 0 or 9 (B)$^\heartsuit$ 1 or 8 (C) 2 or 7 (D) 3 or 6 (E) 4 or 5

Solution. A little long division shows that $\frac{1}{7} = 0.\overline{142857}$, with period 6. So the 100th digit is the same as the 100 mod 6 = 4th digit, which is 8.

Problem 5. What are the leftmost 3 digits of

$$142857143 \cdot 121935032 \quad ?$$

(A) 168 (B) 171 (C) 172 (D) 173 (E)$^\heartsuit$ 174

Solution. You could just start multiplying, but that's tedious, since the usual multiplication algorithm computes the digits from right to left. Instead try to recognize the number 142857143. You should have noticed in Problem 4 that $1/7 = 0.\overline{142857}$, so that 142857143 is close to $10^9/7$; in fact, it is exactly

$$10^9 + 1 = \frac{1000000001}{7}.$$

So the product you're trying to compute is

$$\frac{1000000001}{7} \cdot 121935032 = \frac{121935032121935032}{7}.$$

Now start long division — an algorithm that computes digits left to right !

$$\begin{array}{r@{}l}
 & 174\ldots \\
7\big) & 1219\ldots \\
 & \underline{7} \\
 & 51 \\
 & \underline{49} \\
 & 29
\end{array}$$

Problem 6. Compute

$$\frac{1}{\sqrt{0}+\sqrt{2}} + \frac{1}{\sqrt{1}+\sqrt{3}} + \frac{1}{\sqrt{2}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{2014}+\sqrt{2016}}.$$

(A) $\dfrac{\sqrt{2016}}{2}$ (B) $\dfrac{\sqrt{2016}+\sqrt{2015}}{2}$ (C) $\dfrac{\sqrt{2016}+\sqrt{2014}}{2}$
(D) $\dfrac{\sqrt{2016}+\sqrt{2015}-\sqrt{2014}}{2}$ (E)$^\heartsuit$ $\dfrac{\sqrt{2016}+\sqrt{2015}-1}{2}$

Solution. Since $\frac{1}{\sqrt{n}+\sqrt{n+2}} = \frac{\sqrt{n+2}-\sqrt{n}}{2}$, we have that

$$\sum_{n=0}^{2014} \frac{1}{\sqrt{n}+\sqrt{n+2}} = \frac{1}{2}\left(\sum_{n=0}^{2014}\sqrt{n+2} - \sum_{n=0}^{2014}\sqrt{n}\right);$$

in this last difference between sums, everything cancels except $(\sqrt{2015}+\sqrt{2016}) - (\sqrt{0}+\sqrt{1}) = \sqrt{2016}+\sqrt{2015}-1$.

Problem 7. The roots of the 6th degree polynomial

$$x^6 - 19x^5 + 145x^4 - 565x^3 + 1174x^2 - 1216x + 480$$

are $1, 2, 3, 4, 5$, one of which occurs with multiplicity 2. Which one?

(A) 1 (B) 2 (C) 3 (D)$^\heartsuit$ 4 (E) 5

Solution. There are two simple facts that could be used here:

(a) The sum of the roots of a degree n polynomial with leading coefficient 1 is the negative of the coefficient of x^{n-1}.

(b) The product of the roots of a degree n polynomial with leading coefficient 1 is $(-1)^n$ times the constant term.

To see why these are true, call the roots $r_1, \ldots, r_n$ and try to determine the degree $n-1$ and constant terms of

$$(x-r_1)(x-r_2)\cdots(x-r_n).$$

In this case, we know five roots, 1, 2, 3, 4, 5, and want to know the sixth root r. So (a) gives

$$1+2+3+4+5+r = 19,$$

while (b) gives

$$1\cdot 2\cdot 3\cdot 4\cdot 5\cdot r = 480.$$

Solving either of the two equations gives $r = 4$.

Problem 8. How many numbers n, $0 < n < 10^5$, have the property that any permutation of their digits is divisible by 5?

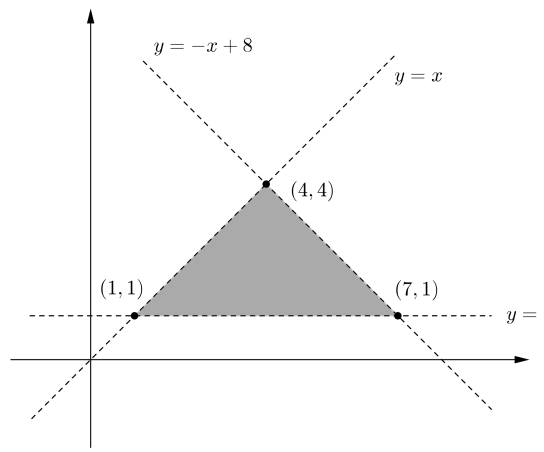
(A) 4 (B) 5 (C)$^\heartsuit$ 31 (D) 83 (E) 1999

Solution. If n is such a number, the only digits allowed for n are 0 and 5. Since n has at most 5 digits, we have $2^5 - 1 = 31$ choices for n. (We have to subtract 1 because not every digit can be zero.)

Problem 9. If you know P is closer to $(3, 1)$ than to $(1, 3)$ and closer to $(2, 4)$ than to $(4, 6)$, and closer to $(3, 2)$ than to $(3, 0)$, what is the area of the region in which P is constrained to lie?

(A)$^\heartsuit$ 9 (B) 18 (C) 27 (D) 0, because there is no such point P
(E) ∞, because the region is unbounded

Solution. The set of points equidistant between two points A, B is the perpendicular bisector of the segment $\overline{AB}$. So the points equidistant from $(3, 1)$ and $(1, 3)$ lie on the line $y = x$. Since P is closer to $(3, 1)$, it must lie below/to the right of this line. Similarly, P must be below/to the left of the line $y = -x + 8$, and above $y = 1$. A quick sketch of these lines shows that P lies in a triangular region with base 6 and height 3, so $A = 9$.



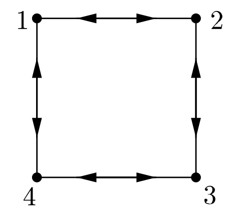
Problem 10. Four friends go on vacation, each to a different location. Each sends a postcard to two friends, chosen at random from the three others with equal probability. What is the probability that each one gets postcards from the same two friends she sent postcards to?

(A) 0 (B) $\frac{1}{81}$ (C)$^\heartsuit$ $\frac{1}{27}$ (D) $\frac{1}{8}$ (E) $\frac{1}{6}$

Solution. Call the friends 1, 2, 3, and 4, and assume 1 sends postcards to 2 and 4, denoted symbolically by $1 \to (2, 4)$. Then $2, 3, 4$ each have 3 options (e.g., $2 \to (1, 3)$, $2 \to (1, 4)$, $2 \to (3, 4)$). So there are 27 possible outcomes. Of these, only 1 has the desired result, namely

$$\begin{aligned} 1 &\to (2, 4), \\ 2 &\to (1, 3), \\ 3 &\to (2, 4), \\ 4 &\to (1, 3). \end{aligned}$$

This can also be represented by a graph:



Problem 11. Seven friends go on vacation, each to a different location. Each sends a postcard to three of the friends, chosen at random from the six others with equal probability. What is the probability that each one gets postcards from the same three friends he sent postcards to?

(A)$^\heartsuit$ 0 (B) $\frac{1}{\binom{6}{3}^4}$ (C) $\frac{1}{\binom{6}{3}^3}$ (D) $\frac{1}{6^3}$ (E) $\frac{1}{6!}$

Solution. If you try to draw a graph to represent this situation, you'll find that you can't. There would be 7 vertices (friends), and two vertices will be connected by an edge if the corresponding friends exchange postcards. So each vertex has degree 3, for a total degree of 21. But the total degree of a graph is always even — it's twice the number of edges ! So there's no way for the friends to exchange postcards in the desired way.

Problem 12. There are 100 lockers in a row, labelled 1-100, and all the doors are initially closed. One hundred mischievous students begin opening and closing doors: the first student opens every door; the second closes every other door, beginning with the first; ..., the nth student changes the status of every nth door, beginning with the first. How many of the doors will be open after the 100th student finishes?

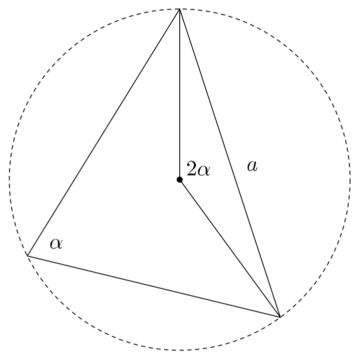
(A) 1 (B)$^\heartsuit$ 9 (C) 10 (D) 11 (E) 50

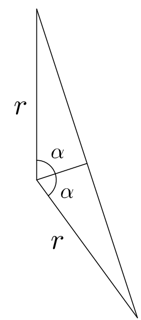
Solution. Notice that the nth student changes the status of doors #1, $n+1$, $2n+1$, $\ldots$, $kn+1$. So the number of times that door N changes equals the number of times $N = kn+1$, i.e., $N-1 = kn$. If $N = 1$, this happens 100 times, once for each of the 100 students, and so door 1 ends up closed. For $N > 1$, door N will end up open precisely when $N-1$ has an odd number of factors. Since every factor a of $N-1$ has a "partner" $b = \frac{N-1}{a}$, we see that $N-1$ has an even number of factors **unless** $N-1$ is the square of a positive integer, i.e., $N-1 = 1, 4, 9, \ldots, 81$. So 9 locker doors are open: # 2, 5, 10, 17, 26, 37, 50, 65, 82.

Problem 13. In a scalene triangle with side lengths $a < b < c$ opposite angles $\alpha < \beta < \gamma$, which of the following is largest?

(A) $\frac{a}{\sin\alpha}$ (B) $\frac{b}{\sin\beta}$ (C) $\frac{c}{\sin\gamma}$ (D) the diameter of the circumscribed circle
(E)$^\heartsuit$ those are all the same

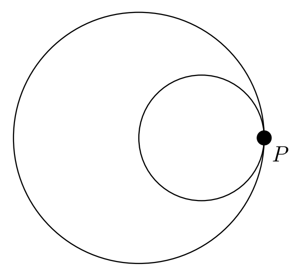
Solution. The law of sines says that the first three are the same. Any time the same number shows up in three different ways, you should look for a geometric meaning. We claim that the common number $a/\sin\alpha = b/\sin\beta = c/\sin\gamma$ is the diameter of the circumscribed circle. To see this, draw the circumscribed circle, then draw the angle from the center to two of the vertices. This angle is twice the corresponding angle in the triangle (first figure). Bisect the double angle (second figure).

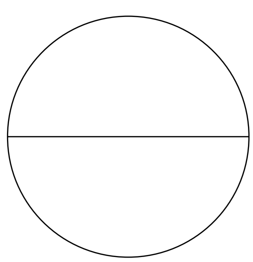


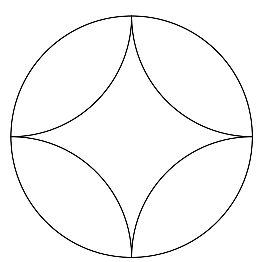


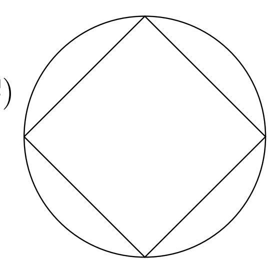
Then $\sin\alpha = \frac{a/2}{r}$, so $2r = a/\sin\alpha$.

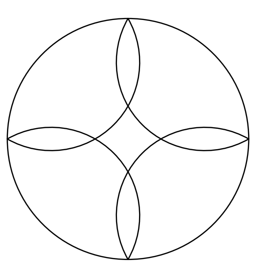
Problem 14. A circle of radius 1 rolls without slipping inside a circle of radius 2. If P is a point on the circumference of the smaller circle, what is the shape of the path that P traces as the inner circle rolls?

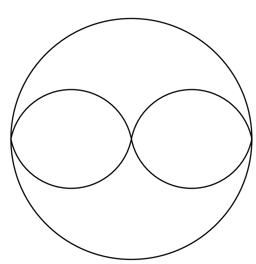


(A)♡ 

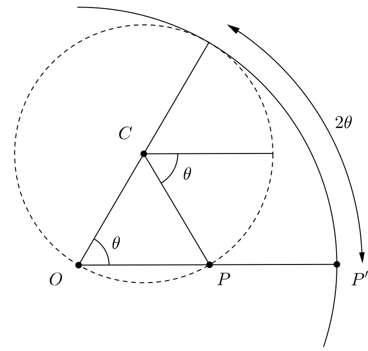
(B) 

(C) 

(D) 

(E) 

Solution. Consider the situation after the inner circle has rolled a short distance, so that the angle formed by its center, C, the center O of the larger circle, and the original location P' of P is θ. The arc length on the circle of radius 2 is 2θ, so the circle of radius 1 has rolled along its circumference a length 2θ. So P has turned through an angle 2θ **clockwise from the horizontal**. Since the smaller circle is rolling on a circle, it has also turned through an angle θ counterclockwise, so P has turned net θ radians clockwise from the horizontal. Now see the diagram. (Note that P clearly passes through the center of the larger circle, so you can easily eliminate b, c, and d.)

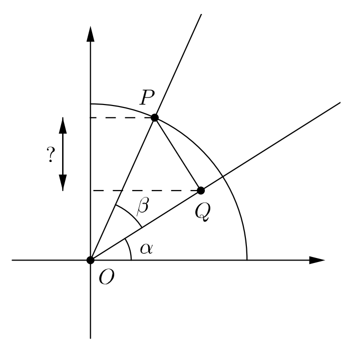


Problem 15. Suppose x and y are two-digit numbers with the property that $\frac{1}{x} = 0.\overline{0y}$ and $\frac{1}{y} = 0.\overline{0x}$. (For example, if $x = 12$, then $\frac{1}{y} = 0.\overline{012} = 0.012012012\ldots$.) What is $x + y$?

(A)$^\heartsuit$ 64 (B) 70 (C) 120 (D) There is more than one pair.
(E) There are no such x and y.

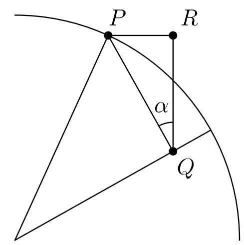
Solution. $\frac{1}{x} = 0.\overline{0y} = \frac{y}{999} \Rightarrow xy = 999$, so 999 must factor as a product of 2 two-digit numbers. Since $999 = 3^3 \cdot 27$, the unique such factorization is $999 = 27 \cdot 37$, and so $x + y = 27 + 37 = 64$.

Problem 16. Two angles α and β are chosen in the first quadrant as shown, together with a circle of radius 1. P lies on the circle, and $PQ \perp OQ$. What is the vertical distance between P and Q?



(A) $\sin(\beta)$ (B) $\sin(\alpha+\beta)$ (C) $\sin(\beta-\alpha)$ (D)$^\heartsuit$ $\sin(\beta)\cos(\alpha)$ (E) $\sin(\alpha)\cos(\beta)$

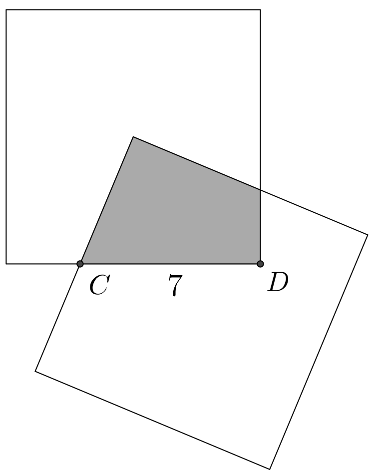
Solution. Draw the auxiliary triangle PQR. Notice that the angle at Q is α, and the hypotenuse PQ is $\sin(\beta)$. So $\cos(\alpha) = \frac{QR}{PQ} = \frac{QR}{\sin(\beta)}$, so $QR = \cos(\alpha)\sin(\beta)$.



Why is this interesting? This is half of the proof of the addition formula for sine. If you also find the height of Q above the x-axis (it's $\cos(\beta)\sin(\alpha)$), then the sum

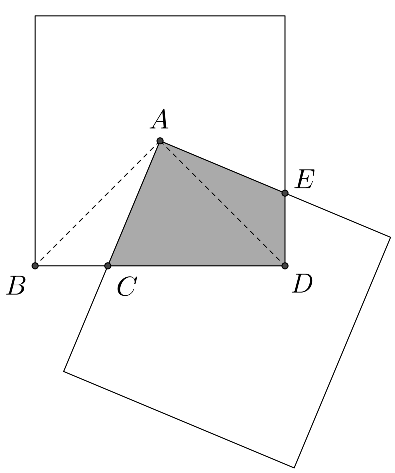
$$\cos(\beta)\sin(\alpha) + \cos(\alpha)\sin(\beta)$$

is the height of P above the x-axis — i.e., $\sin(\alpha + \beta)$.



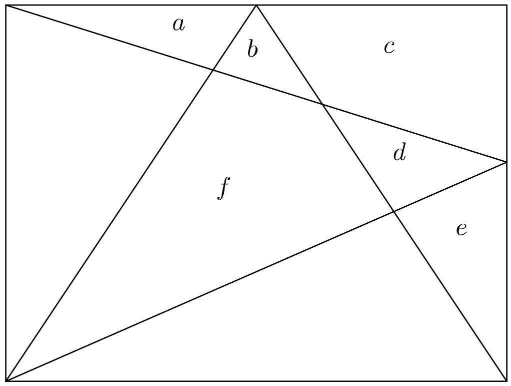
Problem 17. Consider two equal squares of side length 10. Assume a vertex of one of the two squares coincides with the center of the other square. What is the area of the intersection of these squares?

(A) $\frac{100}{3}$ (B) 20 (C) $\frac{10\sqrt{2}}{2}$ (D)$^{\heartsuit}$ 25 (E) $8\sqrt{2}$.



Solution. In the figure, ΔADE is congruent to triangle ΔABC. This is intuitively plausible; to get a proof going, notice that angles DAE and BAC are both complementary to angle CAD, while angle ABC and angle ADE both measure 45°. This is already enough to show ΔADE is similar to ΔABC — but since AB and AD have the same length, the triangles are in fact congruent. It follows that the area of the shaded region is the same as the area of triangle ABD, which is $\frac{1}{2} \cdot 10 \cdot 5 = 25$.

Note that our solution shows the answer is independent of the length of CD !



Problem 18. In the diagram shown, the letters a–f represent the areas of the regions. Which combination of a, b, c, d, e adds up to f?

(A) $a+b+c+d+e$ (B)$^{\heartsuit}$ $a+c+e$ (C) $a+b+d+e$ (D) $b+c+d$
(E) no combination is guaranteed to equal f

Solution. Let p be the area of the unlabelled region adjacent to the left side of the rectangle, and let q be the area of the unlabelled region adjacent to the bottom. Let r be the area of the whole rectangle. Then

$$p+f+d=\frac{1}{2}r,$$
$$q+f+b=\frac{1}{2}r,$$

and

$$a+b+c+d+e+f+p+q=r.$$

Add the first two equations:

$$p+q+2f+d+b=r.$$

Then subtract the third equation:

$$f-a-c-e=0.$$

So $f=a+c+e$.

Problem 19. Think of a positive integer x. Multiply it by 3. Add up the digits. Multiply that by 3. Add the digits one last time. Is your number now 9? Let n be the smallest x such that your number at the end is not 9. What is the sum of the digits of n?

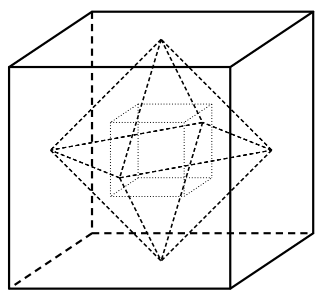
(A) 5 (B) 6 (C) 8 (D) 9 (E)$^{\heartsuit}$ 11

Solution. We work backwards. This process always gives a number divisible by 9. Therefore, the smallest possible final number not equal to 9 is 18. The smallest number whose digit sum is 18 is 99. We now look for the smallest number divisible by 3 whose digit sum is 33. This number is 6999. Dividing by 3 gives us $n=2333$. To see that $n=2333$ is the smallest possible x, read this solution in reverse. (If $x<2333$, then $3x<6999$, so the digit sum is <33, etc...) Thus, the answer is $2+3+3+3=11$.

Problem 20. Start with a cube. Connect the centers of the faces to create an octahedron. Connect the centers of the faces of the octahedron to form another cube. What is the ratio of the volume of the larger cube to the volume of the smaller cube?

(A) 6 (B) 8 (C) 9 (D)$^\heartsuit$ 27 (E) 36

Solution.



For simplicity, assume that the initial cube has vertices at $(\pm1, \pm1, \pm1)$, so the edge length is 2 and the volume is 8. The centers of the faces are $(\pm1, 0, 0), (0, \pm1, 0)$, and $(0, 0, \pm1)$. For the octahedron, the center of each triangular face is the average of its vertices, e.g.,

$$\frac{1}{3}((1,0,0)+(0,1,0)+(0,0,1)) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

Similarly, $(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3})$ is a vertex of the inner cube, which therefore has edge length $\frac{2}{3}$ and volume $\frac{8}{27}$. Finally, the ratio of the larger cube to smaller is $8/\frac{8}{27} = 27$.

Notice that we don't need the volume of the octahedron (which is $\frac{4}{3}$) and that (unlike ciphering problem #3) the ratio of the first volume to the third is **not** the square of the ratio of the first volume to the second.

Problem 21. Let r_1, r_2, r_3, and r_4 be real roots of the polynomial

$$x^8 - 6x^7 - 15x^4 - 6x + 1$$

with $r_1 \le r_2 \le r_3 \le r_4$. What is $r_1r_2 + r_3r_4$?

(A) -5 (B) 0 (C)$^\heartsuit$ 2 (D) 3 (E) 5

Solution. Let $f(x) = x^8 - 6x^7 - 15x^4 - 6x + 1$. Since $f(0) = 1$ while $f(-1) = -1$ and $f(1) = -25$, we know $f(x)$ has two roots a and b such that $-1 < a < 0 < b < 1$. Since $f(x)$ is palindromic, we have $f(x) = x^8 f(1/x)$. Therefore if r is a root, then so is $1/r$. Thus, $1/a$ and $1/b$ are also real roots of $f(x)$.

According to Descartes' rule of signs, the number of positive roots of $f(x)$ is either equal to the number of sign differences beween consecutive coefficients, or smaller than this by an even integer. Applied to our $f(x)$, we see that $f(x)$ has at most two positive roots. Applying the same rule to $f(-x)$ reveals that $f(x)$ has at most two negative roots. So the real roots we found above — $a, b, 1/a, 1/b$ — are all the real roots of $f(x)$. Since

$$\frac{1}{a} < a < 0 < b < \frac{1}{b},$$

we see that $r_1r_2 + r_3r_4 = 1 + 1 = 2$.

Problem 22. We all know that if a and b are the legs of a right triangle and c is the hypotenuse, then $a^2 + b^2 = c^2$. If instead we consider the equation

$$a^{-2} + b^{-2} = x^{-2},$$

what does x represent?

(A) the hypotenuse of the triangle (B) the perimeter
(C)$^\heartsuit$ the altitude (from hypotenuse to opposite vertex) (D) the area
(E) the diameter of the inscribed circle

Solution. Notice that

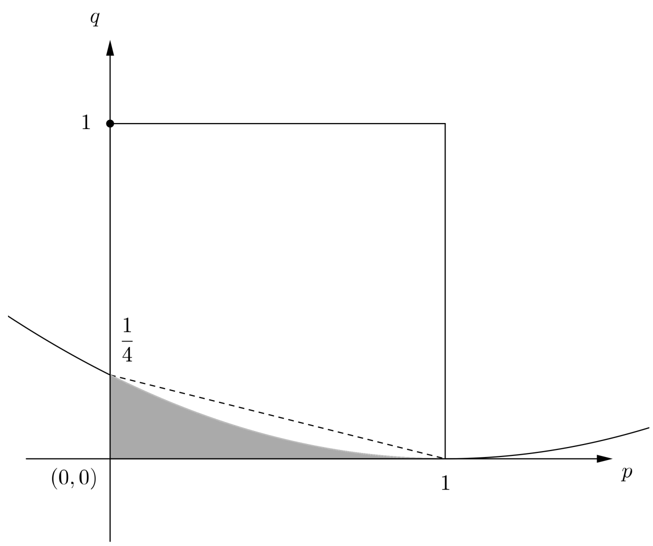
$$\begin{aligned} a^{-2} + b^{-2} = x^{-2} \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{x^2} &\Rightarrow \frac{a^2+b^2}{a^2b^2} = \frac{1}{x^2} \\ &\Rightarrow \frac{c^2}{a^2b^2} = \frac{1}{x^2} \\ &\Rightarrow x^2 = \frac{a^2b^2}{c^2} \\ &\Rightarrow x = \frac{ab}{c} \\ &\Rightarrow cx = ab \\ &\Rightarrow \frac{1}{2}cx = \frac{1}{2}ab = \text{area}. \end{aligned}$$

Thus, x is the altitude from the hypotenuse.

Problem 23. Each point (p, q) in the unit square $0 \le p \le 1$, $0 \le q \le 1$ corresponds to a parabola $y = x^2 + px + q$. Let R be the region in the unit square corresponding those parabolas that intersect the line $y = x$. Which of the following is true about the area A of R?

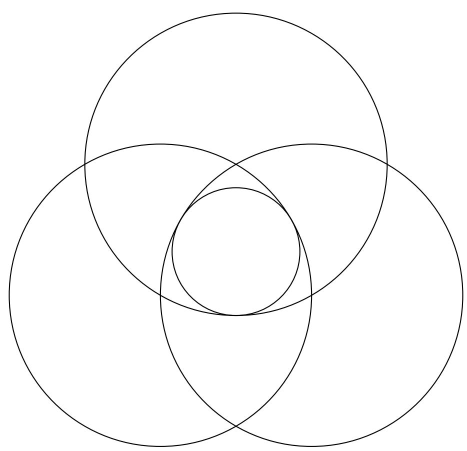
(A)$^\heartsuit$ $0 \le A < \frac{1}{5}$ (B) $\frac{1}{5} \le A < \frac{2}{5}$ (C) $\frac{2}{5} \le A < \frac{3}{5}$ (D) $\frac{3}{5} \le A < \frac{4}{5}$
(E) $\frac{4}{5} \le A \le 1$

Solution. The parabola $y = x^2 + px + q$ intersects $y = x$ if and only if $x^2 + px + q = x$ has a real solution, i.e., $x^2 + (p-1)x + q$ has a real root. This happens when the discriminant $(p-1)^2 - 4q$ is nonnegative. The discriminant is 0 along the parabola $q = \frac{1}{4}(p-1)^2$, which has vertex at $(p, q) = (1, 0)$ and intersects the left-hand side of the unit square at $(p, q) = (0, \frac{1}{4})$. We're interested in the region underneath this parabola. This lies inside the triangle with base 1 and height $\frac{1}{4}$, and so $A < \frac{1}{8} < \frac{1}{5}$.



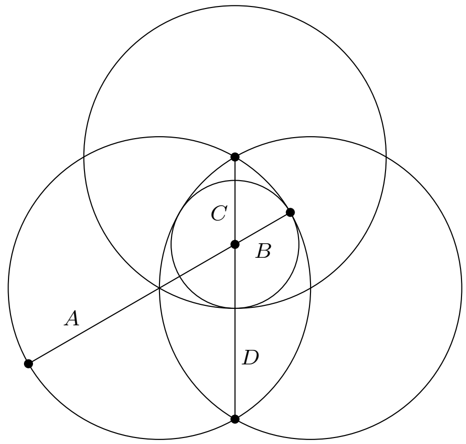
Using integral calculus, we can compute the exact area; it is $\int_0^1 \frac{1}{4}(p-1)^2\,dp = \frac{1}{12}(p-1)^3\big|_0^1 = \frac{1}{12}$. Hence, a random parabola $y = x^2 + px + q$ with $0 \le p, q \le 1$ has a 1 in 12 chance of intersecting the line $y = x$.

Problem 24. Let three circles of radius 1 each intersect the others' centers and draw a small circle internally tangent to all three as in the figure below. What is the radius of the small circle?



(A) $\frac{1}{3}$ (B)$^{\heartsuit}$ $1 - \frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{6}}$ (D) $\frac{\sqrt{5}-1}{3}$ (E) $\frac{1}{6} + \frac{1}{2\sqrt{3}}$

Solution. Let A, B, C, and D be line segments as shown below.



Letting the radius of the small circle be r, we immediately have $|A| = 2 - r$, $|B| = r$, and $|C| = 1 - r$. By symmetry the centers of the large circle form an equilateral triangle with side length 1. The height of this equilateral triangle is $\sqrt{3}/2$, so we have $|C| + |D| = \sqrt{3}$, giving $|D| = \sqrt{3} - 1 + r$.

Now recall the *power of a point theorem*: Given a circle and an arbitrary point P, pass any ray through P intersecting the circle. The product of the distances from P to the points of intersection is independent of the initial ray.

Applied to our situation, with P taken as the center of the small circle, we find that $|A||B| = |C||D|$. Hence,

$$\begin{aligned}(2-r)r &= (1-r)\left(\sqrt{3} - 1 + r\right)\\ &\implies 2r - r^2 = \sqrt{3} - 1 - r\sqrt{3} + 2r - r^2\\ &\implies r\sqrt{3} = \sqrt{3} - 1\\ &\implies r = 1 - \frac{1}{\sqrt{3}}.\end{aligned}$$

Problem 25. Amber the ant and Byron the beetle have homes on opposite vertices of a unit cube. One day Amber and Byron arrange to meet. Both set off from their homes at the same time. Each walks along edges from vertex to vertex at a constant speed of one edge per minute, and each time one reaches a vertex they choose at random a neighboring edge to walk along (the choice might be to return along the edge they just took). What is the expected number of minutes it will take for them to find each other?

(A) 5 (B) 7 (C) 10 (D)$^\heartsuit$ 13 (E) they never meet

Solution. First notice that instead of meeting at a vertex, Amber and Byron can only meet at the middle of an edge (since there will always be an odd number of edges between them when they are both at a vertex). Now we can relate the expected time E to itself. Let E_1 be the expected time it takes to meet from a position where Amber and Byron are separated by a single edge. After the first step, there is a $\frac{1}{3}$ chance that Amber and Byron are again at opposite corners of the cube and a $\frac{2}{3}$ chance that they are now separated by a single edge. This gives

$$E = 1 + \frac{1}{3}E + \frac{2}{3}E_1$$

Now, when Amber and Byron are separated by an edge, there is a $\frac{1}{9}$ chance they both walk along that edge and meet in the middle (in half a minute). If they don't choose the same edge, then of the eight remaining choices, two choices end them back at opposite corners and six choices have them again separated by 1 edge. This gives

$$E_1 = \frac{8}{9}\left(1 + \frac{1}{4}E + \frac{3}{4}E_1\right) + \frac{1}{9} \cdot \frac{1}{2}$$

Solving this system of equations gives $E_1 = \dfrac{23}{2}$ and $E = 13$.

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