1785

Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 13, 2007

WITH SOLUTIONS

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. How many 2-digit numbers are there such that the sum of the digits plus the product of the digits equals the number itself?

(A) 2 (B)♡ 9 (C) 10 (D) 18 (E) None of the above

Solution. We have

$$a + b + ab = 10a + b, \quad \text{or} \quad ab = 9a$$

So the last digit must be 9 and the first digit any number between 1 and 9.

Problem 2. Tyler has in his pocket assorted coins (including possibly pennies, nickels, dimes, quarters, and fifty-cent pieces). What is the largest possible amount of money he can have without being able to make change for a nickel, a dime, a quarter, a fifty-cent piece or a dollar?

(A) \$.74 (B) \$.69 (C) \$.79 (D) \$.94 (E)♡ None of the above

Solution. He can have four pennies, no nickels, four dimes, one quarter, and one fifty-cent piece. This makes a grand total of \$1.19.

Problem 3. What is length of the side of the largest square with one vertex at the origin that will fit inside the parabola $y = x^2$?

(A) 1 (B)♡ $\sqrt{2}$ (C) 2 (D) $\sqrt{3}$ (E) 3

Solution. The square must be symmetric about the y-axis, and so one of its sides must lie along the line $y = x$. This line intersects the parabola at $(1, 1)$, so the square has sides of length $\sqrt{2}$.

Problem 4. What is the locus of points equidistant from a circle and a point in the plane not lying on the circle?

(A) a parabola (B) an ellipse (C) one branch of a hyperbola
(D)$^\heartsuit$ either an ellipse or a branch of a hyperbola (E) either an ellipse or a hyperbola

Solution. Without loss of generality, we take the circle to have the equation $x^2+y^2 = 1$ and the point to be at $(a, 0)$ with $a \geq 0$. The set of points (x, y) equidistant from both is given by

$$\begin{aligned} \left(\sqrt{x^2+y^2} - 1\right)^2 &= (x-a)^2 + y^2 \\ 2\sqrt{x^2+y^2} &= 2ax + (1-a^2)\,, \end{aligned} \tag{$*$}$$

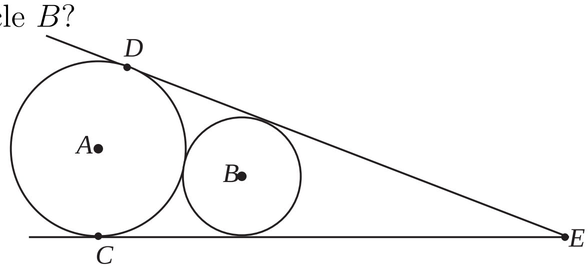
which means the points must satisfy the equation

$$4(1-a^2)x^2 - 4a(1-a^2)x + 4y^2 = (1-a^2)^2\,.$$

When $a < 1$, we get an ellipse, and when $a > 1$, we get a hyperbola. However, we only get one branch of the hyperbola because in equation $(*)$ the values of x must be positive.

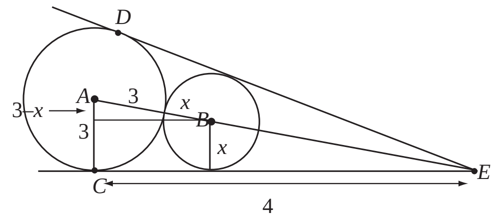
Alternative solution: Let $d_1(x, y)$ be the distance from (x, y) to the point P, and d_2 the distance from (x, y) to the center of the circle.. Let $d(x, y)$ be the distance from (x, y) to the circle (assumed to be of radius r). When (x, y) is in the interior of the circle, we have $d_2 + d = r$, and when (x, y) is in the exterior of the circle, we have $d_2 = d + r$. So, when P is the interior of the circle, we'll have $d_1 + d_2 = r$ (an ellipse), and when P is in the exterior of the circle, we'll have $d_2 - d_1 = r$ (one branch of a hyperbola).

Problem 5. As shown in the diagram, circles A and B are tangent to each other and to the rays $\overrightarrow{EC}$ and $\overrightarrow{ED}$. If the radius of circle A is 3 and $CE = 4$, then what is the radius of circle B?



(A) 1/2 (B) 5/7 (C)$^\heartsuit$ 3/4 (D) 1 (E) None of the above

Solution.



We have $AE = 5$. Next, we obtain from similar triangles the fact that

$$\frac{3-x}{3+x} = \frac{3}{5}, \qquad \text{and so} \quad x = \frac{3}{4}.$$

Problem 6. What is the sum of all 4-digit numbers in which the digits 1,2,3,4 appear exactly once?

(A) 33,330 (B)$^\heartsuit$ 66,660 (C) 133,320 (D) 399,960 (E) None of the above

Solution. There will be $4! = 24$ such numbers, and the average of these numbers will be $(1111 + 4444)/2$. Therefore, the sum is

$$\frac{5555}{2} \cdot 24 = 66,660$$

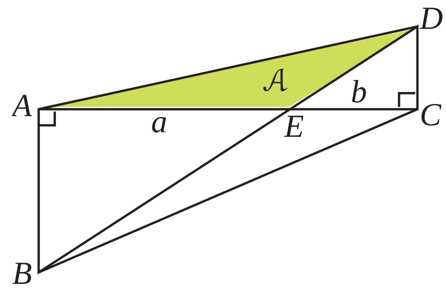
Alternative solution. The numbers occur in complementary pairs, such as 1423 and 4132, each of which sums to 5555. Since there are 12 such pairs, the sum is $12 \cdot 5555 = 66,660$.

Problem 7. How many ways are there to put 8 indistinguishable rooks on an 8-by-8 chess board so that no rook threatens another? (Each rook threatens the rooks which are in the same column or the same row.)

(A) 64 (B)$^\heartsuit$ 40,320 (C) 80,640 (D) 1,625,702,400 (E) None of the above

Solution. The first rook can be put in any of the 8^2 squares. After this, one column and one row are threatened, so the second rook can be put in any of 7^2 squares. For the third rook, we have 6^2 possibilities, etc. Finally, the order of the rooks is unimportant, so we have to divide by $8!$. This gives $(8!)^2/8! = 8!$.

Problem 8. If the area of the shaded triangle $\triangle AED$ is $\mathcal{A}$ and the lengths $AE = a$ and $EC = b$ are given, then the area of quadrilateral $ABCD$ is

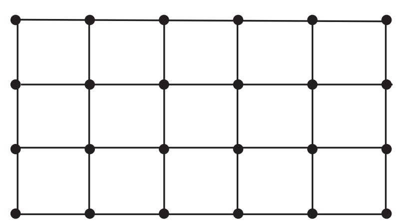


(A)$^\heartsuit$ $\mathcal{A}\dfrac{(a+b)^2}{ab}$ (B) $\mathcal{A}\dfrac{a^2+b^2}{ab}$ (C) $\mathcal{A}\dfrac{ab}{a^2+b^2}$ (D) $\mathcal{A}\dfrac{ab}{(a+b)^2}$
(E) not enough information

Solution. Because $\triangle ABE \sim \triangle CDE$, $AE \cdot CD = CE \cdot AB$, so $\triangle AED$ and $\triangle CEB$ have equal areas $\mathcal{A}$. On the other hand, it is elementary that the area of $\triangle CED$ is $(b/a)\mathcal{A}$ and the area of $\triangle AEB$ is $(a/b)\mathcal{A}$. Thus, the total area of the quadrilateral is

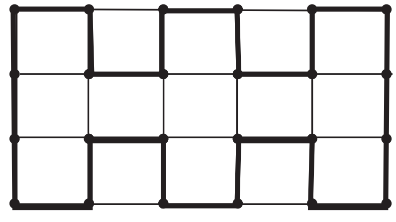
$$\mathcal{A}\left(2+\frac{a}{b}+\frac{b}{a}\right)=\mathcal{A}\frac{(a+b)^2}{ab}.$$

Problem 9. In the rectangular graph pictured below, every edge has length 1. If you draw a Hamiltonian circuit (i.e., a path along the edges that goes through each vertex exactly once, beginning and ending at the same vertex), what is the area of the region it encloses?



(A) 9 (B)$^\heartsuit$ 11 (C) 13 (D) 15 (E) The area depends on the Hamiltonian circuit.

Solution. Pick's Theorem tells us that a polygon (with its vertices only at the vertices of our graph) that encloses i interior vertices and has b vertices on the edges of the polygon will enclose an area $A = b/2 + i - 1$. Since a Hamiltonian circuit must have every vertex on the boundary, it must enclose an area of $24/2 - 1 = 11$. One example is shown below, but there are several others!



Problem 10. The year 2002 was a palindrome: it reads the same forward and backward. The previous palindrome year was 1991. What is the maximal number of non-palindrome years one could have in a row (between 1000 and 9999)?

(A) 10 (B) 11 (C) 79 (D) 81 (E)$^\heartsuit$ 109

Solution. Suppose we now live in a palindrome year $abba$. When is the next palindrome year? Consider two cases:

1. $b = 9$ (i.e. the year is $a99a$, and $a < 9$. Then in 11 years there will be another palindrome year $(a + 1)00(a + 1)$. For example, years 3993 and 4004.
2. $b < 9$. In this case the next palindrome year will be in 110 years: $a(b+1)(b+1)a$. For example, years 9339 and 9449.

Hence, the maximal number of non-palindrome years in a row is 109.

Problem 11. Consider the real solutions of the equation

$$x^4 - 10x^2 + 1 = 0\,.$$

What is the product of all their cubes?

(A) -10 (B) -1 (C)$^\heartsuit$ 1 (D) 10 (E) $10\sqrt{6}$

Solution. Since the polynomial has even degree, the product of all the roots is precisely the constant term, 1. Thus, the product of the cubes is 1. An alternative solution is obtained by noting that the *squares* of the roots are the roots of the quadratic $u^2-10u+1 = 0$, i.e., $u = 5\pm2\sqrt{6}$. So, the roots are $\pm(5+2\sqrt{6})^{1/2}$ and $\pm(5-2\sqrt{6})^{1/2}$, the product of whose cubes is $(5 + 2\sqrt{6})^3(5 - 2\sqrt{6})^3 = \big((5 + 2\sqrt{6})(5 - 2\sqrt{6})\big)^3 = 1$.

Problem 12. The words that can be made by rearranging the letters in the word TOPOLOGY are listed alphabetically. What is the 2007$^{\text{th}}$ word?

(A) GTOLPOYO (B)$^\heartsuit$ OGYOPOLT (C) OGYOPLOT (D) PLOOTOYG
(E) None of the above

Solution. Note first of all that, since there are three O's, we only have $8!/3! = 6720$ possible words. If the first letter is not an O, there are $7!/3! = 840$ possible words; so we see that words starting with G or L account for 1680 words. Thus, we want the 327$^{\text{th}}$ word beginning with O. Now, note that there are $6!/2! = 360$ words starting with OG. The last 60 of these start with OGY. Thus, we need the 27$^{\text{th}}$ word starting with OGY. In increasing order, there are 12 starting with OGYL and 24 starting with OGYO. So we need the 15$^{\text{th}}$ word among the latter. We have 6 starting with

OGYOL, 6 starting with OGYOO, and so we need the third word starting with OGYOP, namely OGYOPOLT.

Problem 13. What is the area of a 13-14-15 triangle?

(A) 80 (B) 82 (C)$^{\heartsuit}$ 84 (D) 86 (E) 88

Solution. If we drop an altitude to the side of length 14, the triangle is split into 5-12-13 and 9-12-15 right triangles. The sum of their areas is $30 + 54 = 84$.

Alternatively, one can use Heron's formula or the law of cosines. For example, using Heron's formula,

$$\begin{aligned} A &= \frac{1}{4}\sqrt{(13+14+15)(13+14-15)(13+15-14)(14+15-13)} \\ &= \frac{1}{4}\sqrt{(42)(12)(14)(16)} = 14 \cdot 6 = 84. \end{aligned}$$

Problem 14. How many zeroes follow the last nonzero digit of 2007!?

(A) 100 (B) 250 (C) 400 (D)$^{\heartsuit}$ 500 (E) None of the above

Solution. We want to know the highest power of 10 dividing 2007!. Each factor of 5 will contribute a factor of 10 (because we come along at least one factor of 2 at the same time), each factor of 25 will contribute a factor of 100 (because we come along at least one factor of 4 at the same time), etc. Letting $[x]$ denote the greatest integer $\leq x$, we need to compute

$$\left[\frac{2007}{5}\right] + \left[\frac{2007}{25}\right] + \left[\frac{2007}{125}\right] + \left[\frac{2007}{625}\right] = 401 + 80 + 16 + 3 = 500\,.$$

Problem 15. Five percent of a group of people are drug-users. Everyone in the group is administered a drug test which is known to be 90% accurate (i.e., 90% of users taking the test are shown to be users, and 90% of nonusers taking the test are shown to be nonusers). If the test indicates that a random person in the group is a user, (approximately) what is the probability that this is a "false positive," i.e., that this result is inaccurate?

(A) 10% (B) 32% (C) 50% (D)$^{\heartsuit}$ 68% (E) 90%

Solution. The probability of a positive test result is 14%: The 5% of the population that are users get a positive test 90% of the time (so a net probability of 4.5%), and the rest of the population get a (false) positive 10% of the time (so a net probability of 9.5%). The chance of a false positive is $9.5/14 \approxeq 67.86\%$.

Problem 16. Suppose n is a positive integer. There is a unique fraction with denominator between 500 and 1000 of the form

$$\frac{n+3}{n^2+7n+5}$$

that is *not* in lowest terms. When we put this fraction in lowest terms, its numerator is

(A) 3 (B)$^\heartsuit$ 4 (C) 5 (D) 7 (E) 8

Solution. In order for the fraction $\dfrac{n+3}{n^2+7n+5}$ *not* to be in lowest terms, the numerator and denominator must have a nontrivial common factor, d. Working mod d, we must have

$$n+3 \equiv 0 \pmod{d} \qquad \text{and} \qquad n^2+7n+5 \equiv 0 \pmod{d},$$

so $n \equiv -3 \pmod{d}$ and $n^2+7n+5 \equiv -7 \equiv 0 \pmod{d}$. So $d = 7$ and $n \equiv 4 \pmod{7}$. In order to get the denominator in the desired range, we must take $n = 25$. The resulting fraction is

$$\frac{28}{805} = \frac{4}{115}.$$

Problem 17. What is the maximal number of rooks that can one put in an 8 by 8 by 8 cube so that no rook threatens another? (Each rook threatens the rooks which are in the same row, column, or vertical.)

(A) 8 (B)$^\heartsuit$ 64 (C) 512 (D) 40,320 (E) None of the above

Solution. Look at the 64 verticals. In each of them there is at most one rook, so the total number can not be more than 64. On the other hand, 64 is possible.

Indeed, assign to every cube its coordinate (x, y, z). Here x, y and z are integers between 1 and 8. Consider all 64 possibilities for the pairs (x, y), and for each of them choose z so that the sum $x + y + z$ is divisible by 8. Then no two are in the same row, column, or vertical. Indeed, if two of them are then two of the coordinates (say x and y) coincide. But then the third must coincide as well since the sum is divisible by 8, and $1 \leq z \leq 8$.

Problem 18. Alex and Meredith play a game beginning with n stones. During each turn, either player can remove any number of stones that is 1 less than a prime number; the player who takes the last stone wins. Assuming Alex and Meredith are equally smart players and Meredith goes first, which value of n guarantees victory for Alex?

(A) 2 (B) 5 (C) 9 (D)$^\heartsuit$ 11 (E) 14

Solution. Alex argues as follows. First, $n+1$ must be composite, since otherwise Meredith will take all the stones on her first move. The smallest integer with this property is $n = 3$. For the next winning value, Alex will need both $n+1$ and $(n-3)+1$ to be composite: without the latter condition, Meredith could take $n-3$ stones, leaving Alex in a losing position. The smallest integer with both these properties is $n = 8$. So, for the next winning value, Alex will need all of $n+1$, $(n-3)+1$, and $(n-8)+1$ to be composite. The smallest integer satisfying these conditions is $n = 11$.

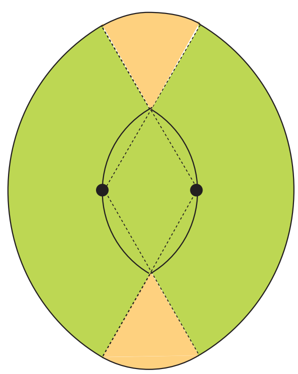
Problem 19. We start with two pieces of rope 1 meter long and two pegs in the ground 1 meter apart. The pieces of rope are knotted together at one end and the other ends are attached to the respective pegs. To the knot another rope of length 1 meter is attached, and at the other end of this rope there is a cow. What is the area (in square meters) of the region on which the cow can graze?

(A) $\dfrac{8\pi}{3}$ (B) $4\pi - \dfrac{\sqrt{3}}{2}$ (C) $\dfrac{16\pi}{3} - \dfrac{\sqrt{3}}{2}$ (D) $6\pi - \dfrac{\sqrt{3}}{2}$ (E)$^\heartsuit$ None of the above

Solution.

The grazing area consists of two 120° sectors of circles of radius 2, which overlap in two equilateral triangles of side 1, and two 60° sectors of circles of radius 1. Hence, the area is

$$2\left(\frac{4\pi}{3} - \frac{\sqrt{3}}{4} + \frac{\pi}{6}\right) = 3\pi - \frac{\sqrt{3}}{2}.$$



Problem 20. In how many distinct ways can one write 1,000,000 as the product of three positive integers? Treat all orderings of the *same* set of factors as one way.

(A)$^\heartsuit$ 139 (B) 196 (C) 219 (D) 784 (E) None of the above

Solution. We have $1,000,000 = 2^6 5^6$. Let the numbers be $2^{a_1}5^{b_1}$, $2^{a_2}5^{b_2}$, and $2^{a_3}5^{b_3}$. Then $a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 6$, and $0 \le a_i \le 6$, $0 \le b_i \le 6$.

First let us count all the ways with repetitions. If $a_1 = 0$ then for $a_2 + a_3 = 6$

there are 7 possibilities. If $a_1 = 1$ then for $a_2 + a_3 = 5$ there are 6 possibilities, etc. In all, we have $7+6+\ldots+1 = 28$ ways. For the pairs (a_i, b_i) thus we have $28^2 = 784$ ways.

Now let us say $a_1 = a_2$ and $b_1 = b_2$. Then $0 \le a_1 \le 3$, so there are 4 ways, and similarly for b's. Together, we have $4^2 = 16$ ways.

There is only one way for all three numbers to be the same: $100 \cdot 100 \cdot 100$.

There are $784 - 1 - 3 \cdot 15 = 738$ triples with three distinct numbers, and we counted them 6 times. There are 15 triples with two numbers coinciding. And there is one triple with all three numbers coinciding. So the answer is

$$\frac{738}{6} + 15 + 1 = 139$$

Problem 21. The harmonic triangle (a variation of Pascal's triangle) has the following properties:

(i) The n^{th} row has n entries, the first and last of which are both $1/n$.

(ii) Each entry is the sum of the two entries immediately *below* it.

The first few rows of the harmonic triangle look like this:

$$\begin{array}{c}
1 \\
1/2 \quad 1/2 \\
1/3 \quad 1/6 \quad 1/3 \\
1/4 \quad 1/12 \quad 1/12 \quad 1/4 \\
1/5 \quad 1/20 \quad 1/30 \quad 1/20 \quad 1/5 \\
\vdots
\end{array}$$

The outermost entries are the terms of the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, which is known to diverge. What is the sum of all the terms in the shaded region?

(A) 1/2 (B)$^\heartsuit$ 1 (C) 7/6 (D) 3/2 (E) The sum is divergent.

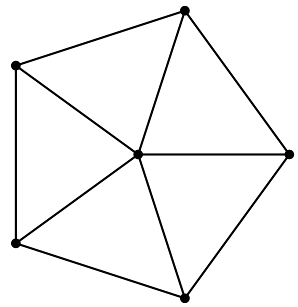
Solution. The sum of the first diagonal of the shaded region is the telescoping series

$$\frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots = \frac{1}{2}.$$

Similarly, the next diagonal sums to 1/6, the next to 1/12, etc. So we find that we want the sum of the entire first diagonal:

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots = 1.$$

Problem 22. A rabbit jumps randomly from vertex to vertex in the following graph, every time choosing the direction randomly.



After a million jumps, what is the probability that the rabbit will be at the center of this graph?

(A) 1/6 (B) 1/5 (C)$^\heartsuit$ 1/4 (D) 1/3 (E) 2/5

Solution. First solution: The probability is proportional to the degree of the vertex (the number of edges coming into the vertex). Hence, the answer is

$$\frac{5}{5+3\cdot 5} = \frac{5}{20} = \frac{1}{4} = 25\%$$

To see why this should be true, note that after many, many jumps, there will be a "stable" distribution of probabilities p_i with which the rabbit can be found at the vertex v_i. These probabilities should satisfy the following system of equations:

$$p_i = \sum_{v_j \text{ is neighbor of } v_i} \frac{p_j}{\deg v_j}$$

Indeed, after the *previous* jump, the rabbit was in one of the neighboring vertices v_j, with probability p_j, and from there it could jump in any of $\deg v_j$ directions, only one of which leads to the vertex v_i.

Now let us see that for any constant c, $p_i = c \deg v_i$ is a solution of the system of equations above:

$$c \deg v_i = \sum_{v_j \text{ is neighbor of } v_i} \frac{c \deg v_j}{\deg v_j} = \sum_{v_j \text{ is neighbor of } v_i} c \quad = c \deg v_i$$

It works! Finally, one must have $\sum p_i = 1$, so $c = \dfrac{1}{\sum \deg v_k}$.

Second solution: Suppose that after many (e.g., 999,999) jumps, the probability that the rabbit is at the center is p. Then the probability of his being at each of the other vertices is $(1-p)/5$. Then the probability of his being at the center on the millionth jump will be

$$0 \cdot p + 5 \cdot \frac{1}{3} \cdot \frac{1-p}{5} = \frac{1-p}{3}.$$

Since this probability must equal p, we have $p = 1/4$.

Problem 23. A student is treated to a four-course meal, presented in random order. If the student eats a course and it is the best course he has eaten at that meal, he exclaims "Wow!" (For example, he will necessarily express delight after the first course.) What is the expected number of times that the student exclaims "wow"? Assume that no two courses are equally good.

(A) 2 (B) 2.5 (C) 3 (D) 3.5 (E)$^\heartsuit$ None of the above

Solution. The probability that the i^{th} course he eats is the best up to that point is $1/i$. Thus, the expected number of exclamations is

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

Problem 24. What is

$$\tan 20^\circ \tan 40^\circ \tan 80^\circ?$$

(A) 1/3 (B) 1 (C) $\sqrt{2}$ (D)$^\heartsuit$ $\sqrt{3}$ (E) None of the above

Solution. First solution: We want

$$\tan 20^\circ \tan 40^\circ \tan 80^\circ = \frac{\sin 20^\circ \sin 40^\circ \sin 80^\circ}{\cos 20^\circ \cos 40^\circ \cos 80^\circ}.$$

Using the double-angle formula for sin three times, we obtain

$$\begin{aligned}\sin 40^\circ \sin 80^\circ \sin 160^\circ &= 8 \sin 20^\circ \cos 20^\circ \sin 40^\circ \cos 40^\circ \sin 80^\circ \cos 80^\circ \\ &= 8(\sin 40^\circ \sin 80^\circ \sin 160^\circ)(\cos 20^\circ \cos 40^\circ \cos 80^\circ)\,.\end{aligned}$$

Thus, the denominator $\cos 20^\circ \cos 40^\circ \cos 80^\circ$ must equal $1/8$. Note that this part appeared in a previous tournament.

For the numerator, we use special angle values and the sum-product formulas $\sin x \sin y = (\cos(x-y) + \cos(x+y))/2$ and $\sin x \cos y = (\sin(x-y) + \sin(x+y))/2$ to get

$$\begin{aligned}\sin 20^\circ \sin 40^\circ \sin 80^\circ &= \frac{1}{2} \sin 20^\circ (\sin 40^\circ - \cos 120^\circ) \\ &= \frac{1}{2} \sin 20^\circ (\cos 40^\circ + \frac{1}{2} \sin 20^\circ) \\ &= \frac{1}{4}(\sin 60^\circ + \sin 20^\circ - \sin 20^\circ) = \frac{\sqrt{3}}{8}\,.\end{aligned}$$

Thus, the original fraction is $\sqrt{3}$.

Second solution: Using the formulas

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}, \quad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta},$$

gives

$$\tan(\alpha+\beta)\tan(\alpha-\beta) = \frac{\tan^2\alpha - \tan^2\beta}{1-\tan^2\alpha\tan^2\beta},$$

Therefore,

$$\begin{aligned}\tan 20^\circ \tan(60^\circ+20^\circ)\tan(60^\circ-20^\circ) &= \tan 20^\circ \frac{\tan^2 60^\circ - \tan^2 20^\circ}{1-\tan^2 60^\circ \tan^2 20^\circ} \\ &= \frac{3\tan 20^\circ - \tan^3 20^\circ}{1-3\tan^2 20^\circ}\end{aligned}$$

Using the formula for the tangent of the triple angle,

$$\tan 3\alpha = \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha},$$

this gives $\tan(3\cdot 20^\circ) = \tan 60^\circ = \sqrt{3}$.

Problem 25. Some 10 points are chosen in the plane. Mark the midpoints of all the intervals with endpoints at these 10 points. What is the smallest number of *distinct* points that can be marked?

(A) 9 (B) 17 (C) 27 (D) 45 (E) None of the above

Solution. If we put the initial 10 points on the x-axis with the coordinates 1,2,...,10 then the midpoints will have integral and half-integral coordinates $1.5 \le x \le 9.5$, and there will be $9+8=17$ such midpoints.

On the other hand, there are always at least 17 distinct midpoints. Indeed, pick two points, say A and B, which are the farthest from each other. Then the midpoints between A and the other 9 points are all distinct and lie in a circle of radius $|AB|/2$ centered at A. Similarly, the midpoints between B and the other 9 points are all distinct and lie in a circle of radius $|AB|/2$ centered at B. This gives $9+9-1=17$ distinct points.

Authors. Written by Boris and Valery Alexeev, Mo Hendon, Tyler Kelly, Alex Rice, and Ted Shifrin.

Sources. Two problems are taken from I.V. Yaschenko, *Invitation to a mathematical festival.* Some problems from Russian olympiads were used as well. One problem was inspired by a recent Putnam problem.