1785

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TEAM ROUND / 1 HOUR / 210 POINTS
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WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

Problem 1 (Divide and conquer)**.** Consider a regular tetrahedron and a sphere with the same centers. Into how many parts can they divide three-dimensional space? The sizes of the tetrahedron and of the sphere can be arbitrary.

Answer. 10

Solution. Fix the size of the tetrahedron. Start with a very small sphere, then gradually enlarge it and see how the number of parts changes. In the beginning, there are 3 parts: the interior of the sphere, the space between the sphere and the tetrahedron, and the outside of the tetrahedron.

When the sphere touches the centers of the sides of the tetrahedron, and then grows a little larger, 4 more parts are added, for a total of 7.

When the sphere touches the centers of the edges, the part between the sphere and the tetrahedron splits into 4 different parts, for a total of 10.

After that, some parts start gluing together, decreasing the total number.

Problem 2 (Deduction)**.** On a 5 by 5 grid, there are 25 squares, and in each square there is a number. For each little square, one computes the sum of the number in this square and the numbers in all the squares that share either a side or a vertex with it. This gives 25 new numbers.

This grid of sums is shown below. How many of the original 25 numbers can be recovered from this grid? In your answer, for each little square, either write down the original number, or put the letter U if the original number is impossible to compute.

18	12	8	10	16
5	1	6	13	17
-12	-4	1	15	7
-19	-19	-12	2	2
-6	-8	-10	-1	1

Answer.

U	U	U	U	U
U	U	U	U	U
U	U	2	U	U
U	U	U	U	U
U	U	U	U	U

Solution. Denote the original 5×5 grid by twenty-five variables $x_{r,c}$ from $x_{1,1}$ to $x_{5,5}$ and their 5×5 grid of sums by twenty-five variables $y_{5,5}$. Then we find that

$$\begin{aligned} y_{1,1} + y_{2,2} - y_{1,2} - y_{2,1} &= (x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2}) \\ &\quad + (x_{1,1} + x_{1,2} + x_{1,3} + x_{2,1} + x_{2,2} + x_{2,3} + x_{3,1} + x_{3,2} + x_{3,3}) \\ &\quad - (x_{1,1} + x_{1,2} + x_{1,3} + x_{2,1} + x_{2,2} + x_{2,3}) \\ &\quad - (x_{1,1} + x_{1,2} + x_{2,1} + x_{2,2} + x_{3,1} + x_{3,2}) \\ &= x_{3,3}, \end{aligned}$$

so we can recover the central entry $x_{3,3}$ from the grid of sums. In our particular case, we find $x_{3,3} = 2$.

To see that no other entry is recoverable, take a grid of xs that produces a given grid of ys. Then adding one to the first and fourth columns while subtracting one from the second and fifth columns of x will not change the grid of ys at all. Another way of putting this as that the grid of sums of the array

+1	−1	0	+1	−1
+1	−1	0	+1	−1
+1	−1	0	+1	−1
+1	−1	0	+1	−1
+1	−1	0	+1	−1

consists entirely of zeroes. It follows that if an entry can be recovered, then it must be in the third column. However, applying the same argument using rows shows that the entry must be in the third row! Thus the central entry is the only entry that can be recovered.

For completeness, we include two grids that have the grid of sums indicated in the problem, but that have only the central element in common:

0	0	0	0	0
0	18	−6	−4	20
0	−13	2	9	−8
0	−17	12	0	−10
0	11	−14	−2	13

1	1	1	1	1
1	15	−7	−3	17
1	−14	2	10	−9
1	−16	13	1	−9
1	8	−15	−1	10

Problem 3 (Ackermann's boxes)**.** Four boxes, numbered from 1 to 4, initially contain one coin each. There are two moves that manipulate the number of coins in each box. The first move allows you to move one coin from box i to box $i+1$, if $1 \le i < 4$; however, if you move a coin from box 3 to box 4, then a "bonus" coin appears in box 4 along with the one that was moved there. The second move allows you to remove one coin from box i and then switch the contents of boxes j and k, if $1 \le i < j < k \le 4$. Of course, at no moment can the number of coins in a box go negative.

What is the largest number of coins that the last box (number 4) could contain after a sequence of such moves?

Answer. 64

Solution. Write (a, b, c, d) for the position with a coins in the first box, b in the second, et cetera. Write $(a, b, c, d) \rightsquigarrow (a', b', c', d')$ if the second position can be obtained from the former. Call the moves in the problem statement the first and second *primitive* moves.

Note that you can go from position (a, b, c, d) to $(a, b, 0, 2c + d)$ by using the first primitive move c times. Call this *doubling*.

Note further that you can go from position $(a,b,0,d)$ to $(a,0,0,2^b d)$ by alternately using the second primitive move and doubling (b times each): $(a,b,0,d) \rightsquigarrow (a,b-1,d,0) \rightsquigarrow (a,b-1,0,2d) \rightsquigarrow (a,b-2,2d,0) \rightsquigarrow (a,b-2,0,2^2 d) \rightsquigarrow \cdots \rightsquigarrow (a,0,0,2^b d)$. If d is zero, then one can go from $(a,b,0,0)$ to $(a,0,0,2^b)$ by first moving over $(a,b,0,0) \rightsquigarrow (a,b-1,1,0) \rightsquigarrow (a,b-1,0,2)$ and then doing the above. Call these sequences of moves *exponentiation.*

Using this shorthand, it is not difficult to obtain $64 = 2^6$ coins:

$$(1,1,1,1) \rightsquigarrow (1,1,0,3) \rightsquigarrow (1,0,0,6) \rightsquigarrow (0,6,0,0) \rightsquigarrow (0,0,0,2^6),$$

using the first primitive move, exponentiation, the second primitive move, and exponentiation again.

In the particular case of the starting position $(1,1,1,1)$, it is possible to explore the tree of possible moves and see that 64 is the maximum number that appears. Instead of writing out that argument (which would be easier), we prove a much more general statement: the maximum number of coins obtainable from position (a,b,c,d) is

$$f(a,b,c,d) = 2^{2^{\cdot^{\cdot^{\cdot^{2^e}}}}},$$

where the tower has a twos and then $e = 2^b \cdot (2c+d)$ on top[1]; in other words, the maximum is the result of iterating the function $x \mapsto 2^x$ a total of a times starting with the initial value $e = 2^b \cdot (2c+d)$. For example, $f(3,4,5,6) = 2^{2^{2^{\left(2^4 \cdot (2 \cdot 5+6)\right)}}}$.

The idea is very straightforward, roughly that at all points it's better to have a higher tower of twos than a higher exponent of two, a higher exponent than multiple of two, and a higher multiple than a constant. The actual proof is a little bit more complicated because of some exceptional cases where the above isn't true.

The proof is by induction on positions, ordered lexicographically. Two technical conditions need to be checked, but will be delegated to a footnote.[2]

[1]Unfortunately, there are a few exceptional cases. If $c = d = 0$, then $e = 2^b$. If $b > 0$, $c = 0$, and $d = 1$, then $e = 2^{b-1} \cdot 3$. If $a > 0$, $b = c = 0$, and $d = 1$, then the tower is one shorter and $e = 3$. And of course $f(0,0,0,0) = 0$.

[2]First, each move decreases the position from the perspective of lexicographic ordering. Second, the lexicographic ordering is a well-ordering, or alternatively, it is impossible to make unboundedly many moves. This is because you can only make a bounded number of moves that involve the first box, and in between, you can only make a bounded number

Our task now is to check that for each position, it is possible to make a move that preserves f, and that no other move increases f. This implies by induction the validity of f as the maximum number of coins obtainable from a given position. We first check that no move increases f: (some of these moves might not be possible if some coordinates are zero)

- $f(a,b,c,d) \geq f(a-1,b+1,c,d)$ because $2^{2^b(2c+d)} \geq 2^{b+1}(2c+d)$,
- $f(a,b,c,d) \geq f(a,b-1,c+1,d)$ because $2 \cdot (2c+d) \geq 2(c+1)+d$ unless we are in the exceptional cases described in the footnote,
- $f(a,b,c,d) \geq f(a,b,c-1,d+2)$ because $2c+d \geq 2(c-1)+(d+2)$ (in fact, equality holds),
- $f(a,b,c,d) \geq f(a-1,c,b,d)$ because $2^{2^b \cdot (2c+d)} \geq 2^c \cdot (2b+d)$,
- $f(a,b,c,d) \geq f(a-1,d,c,b)$ because $2^{2^b \cdot (2c+d)} \geq 2^d \cdot (2c+b)$,
- $f(a,b,c,d) \geq f(a,b-1,d,c)$ because $2 \cdot (2c+d) \geq 2d+c$,
- $f(a,b,c,d) \geq f(a-1,b,d,c)$ because of transitivity: $f(a,b,c,d) \geq f(a-1,b+1,c,d) \geq f(a-1,b,d,c)$ by the first and last of the cases above.

In verifying the above inequalities, which can be a bit complicated at times (especially because of the exceptional cases in the footnote), it helps to use the basic inequalities $2^x \geq x+1$, $2^x \geq 2x$, $x \cdot y \geq x+y$ if $x \geq 2$ and $y \geq 2$, and $x^y \geq x \cdot y$ if $x \geq 2$.

We now check that it is always possible to make a move that preserves f:

- If $a=b=c=0$, then there are no possible moves and we check $f(0,0,0,d)=d$ as the base case.
- If $c>0$, we check $f(a,b,c,d)=f(a,b,c-1,d+2)$.
- If $b>0$ but $c=d=0$, we check $f(a,b,0,0)=f(a,b-1,1,0)$.
- If $a>0$ but $b=c=d=0$, we check $f(a,0,0,0)=f(a-1,1,0,0)$.

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of moves that involve the second box, et cetera. The important property here is that after no more moves are made to the left of a box, the number of coins in that box is a non-increasing, non-negative integer and so is eventually constant.

- If $d = 1$ and $c = 0$ and $b > 0$, we check $f(a, b, 0, 1) = f(a, b-1, 1, 1)$.
- If $d = 1$ and $b = c = 0$ and $a > 0$, we check $f(a, 0, 0, 1) = f(a-1, 1, 0, 1)$.
- If $d > 1$ and $c = 0$ and $b > 0$, we check $f(a, b, 0, d) = f(a, b-1, d, 0)$.
- If $d > 1$ and $b = c = 0$ and $a > 0$, we check $f(a, 0, 0, d) = f(a-1, d, 0, 0)$.

The answer to this problem should be a little surprising. There are two rules, one of which decreases the number of coins by one and the other which *sometimes* increases the number of coins, but always by at most one. If you never use the rule that decreases the number of coins, then there will only be 7 coins at the end. But if you are smarter, then you can get 64 coins.

The result is even more amazing if you start with the position $(4, 0, 0, 0)$ in which case using the first rule gives 8 coins while being smarter gives $2^{2^{2^2}} = 2^{16} = 65536$ coins. The situation only gets more amazing the more boxes there are. If there are five boxes, then the order of growth of f no longer simply involves *towers* of twos, but *iterated* towers of twos.

This general behavior is often called Ackermann-like, after the Ackermann function $A(m, n)$. Much like the behavior in this problem, $A(2, n)$ grows linearly, $A(3, n)$ grows exponentially, $A(4, n)$ grows like a tower of twos, and so on. The definition of the function is that $A(m, n)$ takes two non-negative integer arguments and returns an integer according to

$$A(m,n) \begin{cases} n+1 & \text{if } m = 0, \\ A(m-1, 1) & \text{if } m > 0 \text{ and } n = 0, \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0. \end{cases}$$

Note that like the situation with the boxes, the only arithmetic operations involved are subtraction and addition of one. The interested reader should look up the Ackermann function on Wikipedia for more information about fast-growing functions.

Authors. All problems and solutions are written by Boris and Valery Alexeev.

Sources. Problems 1 and 2 are based upon problems from the Internet Carousel, a Russian, multi-subject online competition available at `http://karusel.desc.ru/` (page in Russian). Problem 3 was based on IMO

2010 Problem 5 by Hans Zantema. The problem was modified to be more suitable for this contest and also so that the answer is neater. (The IMO problem did not require an exact answer.)