1785

Sponsored by: UGA Math Department and UGA Math Club

CIPHERING ROUND / 2 MINUTES PER PROBLEM
NOVEMBER 8, 2014

WITH SOLUTIONS

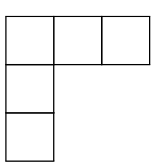
Problem 1. All 4 digit numbers with no repeated digits are listed in increasing order, so that 1234 is first and 9876 is last. How many numbers appear before 2014 in this list? The numbers are in base 10 and have no leading zeros.

Answer. 505 (numbers)

Solution. There are $9 \cdot 8 \cdot 7 = 504$ numbers beginning with 1, and the first two numbers beginning with 2 are 2013 and 2014.

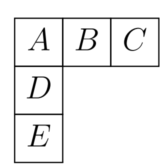
It was brought to our attention that the statement of the problem requires amendment: 1023 is the smallest four digit number with no repeated digits, not 1234. If the list is assumed to begin at 1234 instead of 1023, then none of the numbers $10XY$ appear. Since there are $8 \cdot 7 = 56$ such numbers, this interpretation of the problem leads to an answer of $505 - 56 = 449$. That answer also received full credit.

Problem 2. How many ways can you put the numbers $1, 2, 3, 4, 5$ in the five boxes shown so that the numbers in the top row increase from left to right, and the numbers in the left column increase from top to bottom?



Answer. 6 (ways)

Solution. Clearly 1 has to be in box A, and 5 has to be in box C or E. By symmetry, we need only count the solutions with 5 in box C. This leaves 3 choices for B: 2, 3, or 4, after which the remaining numbers are forced into D and E in increasing order. This gives 3 solutions; reflection gives 3 more.



Problem 3. Find the smallest positive solution to $\sin(x) = \sin(x + \frac{\pi}{6})$.

Answer. $\dfrac{5\pi}{12}$

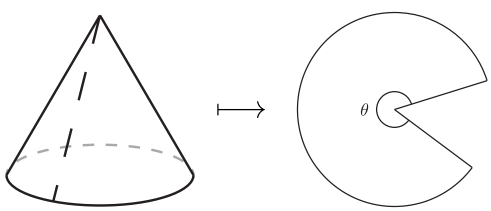
Solution. $\sin(x)$ is symmetric about $x = \frac{\pi}{2}$: $\sin(\frac{\pi}{2} - a) = \sin(\frac{\pi}{2} + a)$. If $x = \frac{\pi}{2} - a$ and $x + \frac{\pi}{6} = \frac{\pi}{2} + a$, then $2x + \frac{\pi}{6} = \pi$, so $x = \frac{5\pi}{12}$.

Problem 4. On the written test you'll take later today, you'll have 25 questions. You'll get 10 points for each problem answered correctly, 2 points for each question left unanswered, and 0 points for each question answered incorrectly. How many different scores are possible?

Answer. 120 (scores)

Solution. There are 26 possible scores ending in 0: $0, 10, \ldots, 250$. There are 25 possible scores ending in 2: $2, 12, \ldots, 242$. There are 24 possible scores ending in 4: $4, 14, \ldots, 234$. There are 23 possible scores ending in 6: $6, 16, \ldots, 226$. There are 22 possible scores ending in 8: $8, 18, \ldots, 218$.

Problem 5. Begin with a cone whose radius equals its height. Cut it open and roll it flat to form a pacman shape. What is the measure of the angle θ in radians?

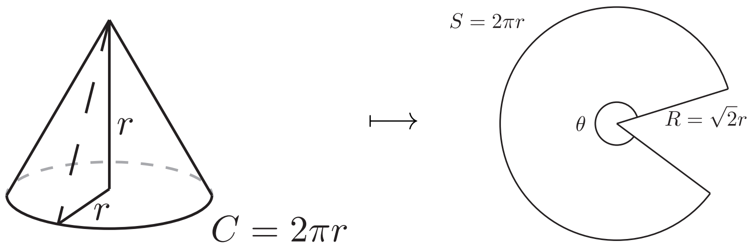


Answer. $\dfrac{2\pi}{\sqrt{2}}$ or $\sqrt{2}\pi$

Solution. Since $r = h$, the slant height of the cone is $\sqrt{2}r$. This is the radius R of the pacman (which is a sector of a circle). The circumference of the base of the cone is $2\pi r$, and this is the arclength S of the sector. Since $S = R\theta$,

$$2\pi r = \sqrt{2}r\theta.$$

So $\theta = \frac{2\pi}{\sqrt{2}} = \sqrt{2}\pi$.

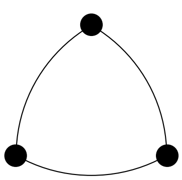


Problem 6. The digital root of a positive integer is obtained by summing its decimal digits, then the decimal digits of the result, and repeating the process until one is left with a single digit number. What is the digital root of 2^{2014}?

Answer. 7

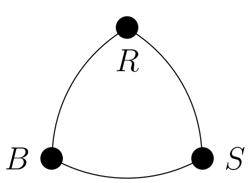
Solution. Every natural number is congruent to its sum of decimal digits modulo 9. Iterating this observation shows that the digital root of n is the unique number among $1, 2, 3, \dots, 9$ congruent to n modulo 9. Using that $2^6 \equiv 1 \pmod 9$, we find that $2^{2014} = (2^6)^{335} \cdot 2^4 \equiv 2^4 \equiv 7 \pmod 9$.

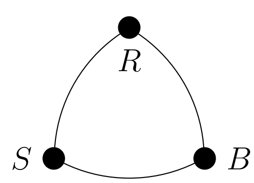
Problem 7. Amber wants to decorate a necklace with 3 colored beads: She has 3 red beads, 3 black beads, and 3 silver beads. How many different ways can she decorate the necklace?



Answer. 10 (ways)

Solution. If all three beads are the same color, there are 3 options. If exactly two are the same color, she has 6 options: 3 choices for the color used twice, followed by two choices for the other color. If she uses 3 different colors, there is only one option.

Notice that 

and 

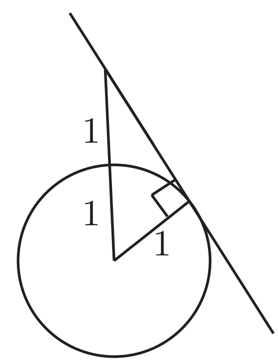
are not "the same" by rotation, but they are by reflection — simply turn the necklace over.

Remark: To find the number of possible necklaces made with three beads, you needed to consider possible actions performed on a necklace that would preserve the necklace.

The set of all such moves forms an algebraic structure called a **group**. In general, the set of all permutations of n objects forms the **symmetric group** S_n. One can solve the necklace problem by viewing S_3 as acting on the set of all possible necklaces and applying a result from group theory known as **Burnside's lemma**. While this approach is overkill in the simple example given above, Burnside's lemma applies in many cases where direct counting is computationally infeasible.

Problem 8. Erect a pole of length 1 perpendicular to the surface of a sphere of radius 1, then shine a light so that the shadow of the pole on the sphere is as long as possible. How long is the shadow?

Answer. $\dfrac{\pi}{3}$



Solution. The longest shadow will be one that reaches the 'horizon'; i.e., the light shines across the top of the pole and touches the sphere tangentially. The resulting triangle has a leg of length 1 and a hypotenuse of length 2, so the central angle is $\arccos(\frac{1}{2}) = \frac{\pi}{3}$.

This question was inspired by an unknown man with a crew cut sitting under the reading light in an airplane seat in front of one of the authors.

Problem 9. Seven years ago, my cat was 4 times as old as my dog. Six years ago, my cat was 3 times as old as my dog. How long ago was my cat twice as old as my dog?

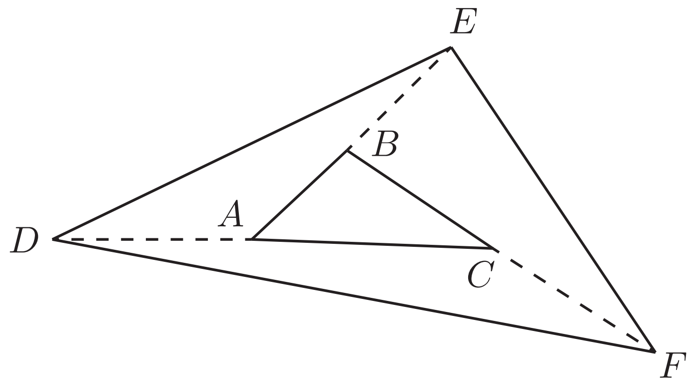
Answer. 3 (years ago)

Solution. If C and D are the current ages of my cat and dog, then

$$C - 7 = 4(D - 7),$$
$$C - 6 = 3(D - 6).$$

Subtracting, we find $-1 = D - 10$, so $D = 9$ and $C = 15$. We want x to satisfy $15 - x = 2(9 - x)$, so $x = 3$.

Problem 10. Start with a triangle ΔABC. Extend AB (in the B direction) until its length doubles. Do the same with BC (in the C direction) and CA (in the A direction). Connect the new endpoints of the extended sides to form a new triangle ΔDEF. If the area of ΔABC is 1, what is the area of ΔDEF?



Answer. 7

Solution. Notice that the base of ΔBEF has the same length as that of ΔABC, since $|AB| = |BE|$, and has twice the height, since $|BF| = 2|BC|$. So the area of ΔBEF is 2. Similarly, both ΔCFD and ΔADE have area 2. So the total area is $2 + 2 + 2 + 1 = 7$.

Authors. Problem #10 is due to Tony Gonzalez. The remaining problems were written by Mo Hendon, Paul Pollack, and Amber Russell.