Sponsored by: UGA Math Department and UGA Math Club

Ciphering Round / 2 minutes per problem
October 2, 2010

WITH SOLUTIONS

No calculators are allowed on this test. 2 minutes per problem, 10 points for each correct answer.

Problem 1. Suppose *distinct* integers a, b, c, and d are chosen between 1 and 9 inclusive. What is the largest *integer* that $\dfrac{a+b}{c+d}$ can be? (Remark: "Distinct" means that no two are the same.)

Answer. 5

Solution. We want to make the numerator as large as possible and the denominator as small as possible, so we take $c+d = 1+2 = 3$. Now we need the largest numerator that is a multiple of 3. This is $a + b = 8 + 7 = 15 = 9 + 6$. The largest possible quotient is 5. (A denominator of 4 or larger does not allow any larger quotient.)

Problem 2. We are given two concentric circles. Each chord of the larger circle that is tangent to the smaller circle is 6 in long. What is the area of the ring between the two circles?

Answer. 9π

Solution. If the radii of the smaller and larger circle are r and R, respectively, then the Pythagorean Theorem gives $R^2 = r^2 + 3^2$, so $\pi(R^2 - r^2) = 9\pi$.

Alternatively, since a solution exists, one may simply assume that the inner circle is just a point. In that case, there is one circle with diameter 6 and thus area 9π.

Problem 3. Two points are picked at random on the circle $x^2 + y^2 = 1$. What is the probability that the chord they determine is longer than 1?

Answer. 2/3

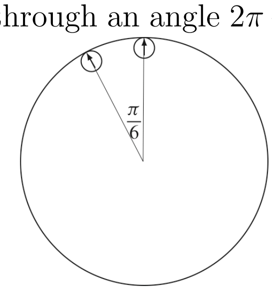
Solution. In order for the chord to have length > 1, it must subtend a central angle θ with $\pi/3 < \theta \leq \pi$. This accounts for 2/3 of the circle.

Problem 4. A circle of radius 1 inch rolls *inside* a circle of radius 12 inches. How many full revolutions does it make before returning to its original position?

Answer. 11

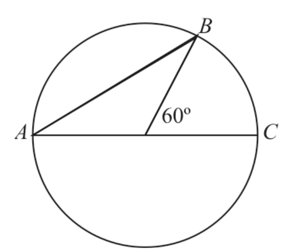
Solution. Imagine cutting the large circle open and straightening it out into a line segment $2\pi \cdot 12$ inches long. Since the radius of the little circle is 1 inch, it makes 12 revolutions when you roll it along the line segment. However, one of those revolutions is made for it when the line segment is rolled back into the large circle. So the inner circle makes only 11 revolutions.

More concretely, imagine the smaller circle with an arrow pointing toward the original point of contact at noon. When the little circle rolls a distance of 2π (the circumference of the smaller circle) around the inside of the larger circle, the arrow will again point toward the point of contact of the two circles. At this stage, the smaller circle has moved through an angle of $\theta = 2\pi/12 = \pi/6$ around the center of the larger circle. Note, however, that the smaller circle has not yet completed one full revolution: It has only rotated through an angle $2\pi - \pi/6 = 11\pi/6$ around its center,



which is 11/12 of a revolution. Thus, as the smaller circle goes through an angle of 2π around the center of the larger circle, it repeats this process 12 times and makes 11 full revolutions around its own center.

Problem 5. How many hours does it take Rachel the Rower to oar from point A to point B if the diameter of the lake is $2\sqrt{3}$ miles and she rows 3 mph? (In the picture, $\overline{AC}$ is a diameter.)



Answer. 1

Solution. If Rachel rows a distance d miles, then the time it takes her will be $d/3$ hours. We can calculate d either by the law of cosines ($d = \sqrt{3}\sqrt{1+1-2\cos(120^\circ)} = 3$) or by dividing the triangle in half and noting that $d = 2(\sqrt{3})\sin(120^\circ/2) = 2\sqrt{3}\sin(60^\circ) = 3$. Even easier, note that $\triangle ABC$ is a right triangle and $\triangle OBC$ is an equilateral triangle (where O is the center of the circle), so $d^2 = (2\sqrt{3})^2 - (\sqrt{3})^2 = 9$, as before.

Problem 6. How many different primes appear as entries in the first 20 rows of Pascal's triangle?

Answer. 8

Solution. Obviously, the primes 2, 3, 5, 7, 11, 13, 17, and 19 appear in their respective rows. No larger prime can appear since no larger prime can even divide the numerator of

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k!}$$

when $n \leq 20$.

Problem 7. $\dfrac{1}{\log_2 120} + \dfrac{1}{\log_3 120} + \dfrac{1}{\log_4 120} + \dfrac{1}{\log_5 120} =$

Answer. 1

Solution. Since $120 = 5!$, we have $\ln 120 = \ln 2 + \ln 3 + \ln 4 + \ln 5$. Since $\log_b a = \dfrac{\ln a}{\ln b}$ (recall that $b^c = \left(e^{\ln b}\right)^c = e^{c \ln b}$), we have $\dfrac{1}{\log_b 120} = \dfrac{\ln b}{\ln 120}$, and

$$\frac{1}{\log_2 120} + \frac{1}{\log_3 120} + \frac{1}{\log_4 120} + \frac{1}{\log_5 120} + = \frac{\ln 2 + \ln 3 + \ln 4 + \ln 5}{\ln 120} = 1\,.$$

Problem 8. How many integers between 1 and 2010 inclusive are divisible by neither 3 nor 5?

Answer. 1072

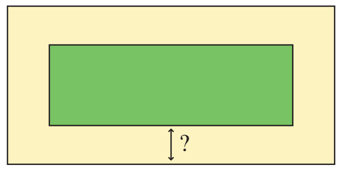
Solution. Note that 2010 is divisible by both 3 and 5. We have $2010/3 = 670$ multiples of 3, $2010/5 = 402$ multiples of 5, and $2010/15 = 134$ multiples of both. Thus, we have

$$2010 - 670 - 402 + 134 = 1072$$

numbers divisible by neither.

Alternatively, notice that 8 of the numbers from 1 to 15 are divisible by neither 3 nor 5. Thus, since 2010 happens to be divisible by 15, we conclude that 8/15 of the numbers from 1 to 2010 are divisible by neither 3 nor 5. This yields $2010 \cdot \frac{8}{15} = 1072$.

Problem 9. Mildred prefers her brownies from the center of the pan, and Millicent prefers them from around the edge. If they bake a 9×12 pan of brownies, how far from the edges of the pan should they cut so that each get equal areas of brownies?



Answer. 3/2

Solution. Let x be the distance from the edge of the pan that we cut. We must solve $(12-2x)(9-2x) = 12 \cdot 9/2$. This gives us $2x^2 - 21x + 27 = (2x-3)(x-9) = 0$, so $x = 3/2$ or $x = 9$. We discard the latter solution.

Problem 10. The number

$$(8\ 1\ 1\ \_)_{\text{nine}}$$

(where this is a base 9 numeral) is a perfect square. What must the last digit be?

Answer. 7

Solution. By considering all the squares mod 9, we see that the last digit of a perfect square must be 0, 1, 4, or 7 (i.e., $0^2 \equiv 3^2 \equiv 6^2 \equiv 0$, $1^2 \equiv 8^2 \equiv 1$, $2^2 \equiv 7^2 \equiv 4$, $4^2 \equiv 5^2 \equiv 7$, all mod 9). On the other hand, the squares mod 8 can be only 0, 1, or 4 (i.e., $0^2 \equiv 4^2 \equiv 0$, $1^2 \equiv 3^2 \equiv 5^2 \equiv 7^2 \equiv 1$, $2^2 \equiv 6^2 \equiv 4$, all mod 8). Now, if N is a base 9 numeral, casting out eights tells us that N is congruent mod 8 to the sum of its digits. This works exactly the same way casting out nines works with base 10 numerals: Since $9 \equiv 1 \pmod 8$, $\sum_{i=0}^{d} a_i 9^i \equiv \sum_{i=0}^{d} a_i \pmod 8$. Only with last digit 7 do we get the sum of the digits to be on the allowed list. In fact, $(85_{\text{nine}})^2 = 8117_{\text{nine}}$.

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