Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 20, 2012

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

   where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit quietly.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. Calculate the sum

$$-1-2-3+4+5+6-7-8-9+10+11+12-\cdots+2008+2009+2010-2011-2012\,.$$

(A) -1017 (B) -1008 (C) -992 (D) -216 (E) None of the above

Problem 2. Frank owes \$5000 and Michael owes \$3000. If Frank had 2/3 of Michael's money in addition to his own, he could exactly pay all his debts; if Michael had 1/2 of Frank's money in addition to his own, he could pay all but \$100 of his debts. What is the total amount of money Frank and Michael have?

(A) \$4000 (B) \$4600 (C) \$5200 (D) \$5250 (E) None of the above

Problem 3. A circle of radius 6 has its center on a circle of radius 5. How far apart are the two points of intersection?

(A) $3\sqrt{2}$ (B) $24/5$ (C) 5 (D) $5\sqrt{2}$ (E) $48/5$

Problem 4. One pipe can fill a tank in 45 minutes and another can fill it in 30 minutes. If these two pipes are open and a third pipe is draining water from the tank, it takes 27 minutes to fill the tank. What is the time, in minutes, that it takes the third pipe alone to drain a full tank?

(A) 48 (B) 54 (C) 60 (D) $64\frac{1}{2}$ (E) None of the above

Problem 5. A sphere is inscribed in a truncated right circular cone (so that it is tangent at the top, the bottom, and along the lateral surface of the cone). If the radii at the top and bottom are 1 and 9, what is the radius of the sphere?

(A) $\sqrt{5}$ (B) 3 (C) $2\sqrt{5}$ (D) $9/2$ (E) 5

Problem 6. Two real numbers are chosen at random between 0 and 10. What is the probability that their sum is greater than 8?

(A) 32% (B) 55% (C) 60% (D) 62.8% (E) 68%

Problem 7. Let A, B, and C be the real numbers defined by

$$A = \sqrt{2+\sqrt{2+\sqrt{2+\dots}}}$$

$$B = \sqrt{6+\sqrt{6+\sqrt{6+\dots}}}, \text{ and}$$

$$C = \sqrt{12+\sqrt{12+\sqrt{12+\dots}}}.$$

Find $A+B+C$.

(A) 6 (B) 9 (C) 3π (D) $5+2\sqrt{5}$ (E) $7\sqrt{2}$

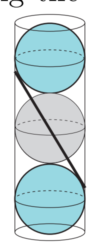
Problem 8. A regular hexagon is inscribed in the unit circle and weights 1, 2, 3, 4, 5, 6 are placed *consecutively* at the vertices. How far from the center of the circle is the center of mass?

(A) 0 (B) 1/4 (C) 2/7 (D) $\sqrt{3}/4$ (E) 3/7

Problem 9. Let $A = \{0, 1, 2, 3, 5, 8, 13, 21, 34, 55\}$. Note that the nonzero numbers in A are consecutive Fibonacci numbers. Define $A+A+A = \{a+b+c : a, b, c \in A\}$ (note that a, b, c are not required to be distinct). What is the smallest positive integer that is not in $A+A+A$?

(A) 20 (B) 33 (C) 54 (D) 88 (E) 166

Problem 10. Instead of putting three tennis balls of radius 1 in a can, a mathematician's pencil (i.e., a line segment) is inserted in place of the middle ball in such a way that when it is tangent to both remaining balls (with both ends touching the can), the top ball is at its usual height. How long is the pencil?



(A) 3 (B) $2\sqrt{3}$ (C) 7/2 (D) 4 (E) $4\sqrt{3}$

Problem 11. How many pairs of rational numbers (a, b) are there for which

$$(a+bi)^7 = a - bi\,?$$

(Remember that $i^2 = -1$.)

(A) 2 (B) 4 (C) 5 (D) 8 (E) 9

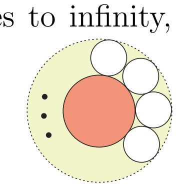
Problem 12. Suppose θ and ϕ are real numbers for which

$$\sin(\theta) + \sin(\phi) = 1/2 \qquad \text{and} \qquad \cos(\theta) + \cos(\phi) = -1/2\,.$$

What is the value of $\sin(\theta + \phi)$?

(A) -1 (B) $-1/2$ (C) 0 (D) $\sqrt{3}/2$ (E) 1

Problem 13. N congruent circles are packed tightly around a circle of radius 1 and inside a concentric circle (which shrinks as N gets bigger). As N goes to infinity, what fraction of the area of the outer ring is covered by the circles?



(A) $\pi/6$ (B) 3/4 (C) $\pi/4$ (D) $3\pi/10$ (E) 1

Problem 14. For each number $n > 1$, let

$$s_n = \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \dots\,.$$

What is the value of the infinite sum $s_2 + s_3 + s_4 + \dots$?

(A) 2 (B) $\pi^2/6$ (C) e^π (D) infinite (E) None of the above

Problem 15. How many subsets of $\{1, 2, 3, 4, \ldots, 10\}$ contain no pair of consecutive integers?

(A) 105 (B) 144 (C) 256 (D) 512 (E) None of the above

Problem 16. The future UGA math department has offices numbered 1 through 2012. A madman breaks into the building and scrawls a 1 on the markerboard outside each office. He then turns around and writes a 2 on the markerboard of each *even-numbered* office. Turning around again, he puts a 3 on those whiteboards whose office numbers are multiples of 3. He continues the same process for 2009 more steps. Finally, he leaves the building and turns himself in to the police.

When the madman is gone, FBI consultant Charlie Eppes totals the numbers on each whiteboard. On how many whiteboards does Charlie find an even sum?

(A) 44 (B) 1024 (C) 1937 (D) 1998 (E) None of the above

Problem 17. A regular hexagon is inscribed in the unit circle and weights 1, 2, 3, 4, 5, 6 are placed in *some* order at the vertices. What is the maximum distance from the center of the circle that you can arrange the center of mass to be?

(A) $2/7$ (B) $1/3$ (C) $4\sqrt{3}/21$ (D) $2\sqrt{13}/21$ (E) $\sqrt{57}/21$

Problem 18. How many times do you expect to have to roll a fair die in order to get each of the numbers one through six to appear?

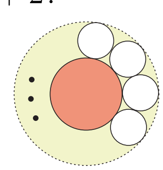
(A) 6 (B) 10 (C) 12 (D) 15 (E) None of the above

Problem 19. Let d_n be the final digit before the decimal point of $(2 + \sqrt{5})^n$. For example, $d_3 = 6$, since $(2 + \sqrt{5})^3 = 76.0131\ldots$. Find $d_1 + d_2 + d_3 + \cdots + d_{2012}$.

(A) 9054 (B) 9060 (C) 10060 (D) 11066 (E) None of the above

Problem 20. For how many positive integers n can we fit tightly packed congruent circles of radius 1 in the ring between concentric circles of radii n and $n + 2$?

(A) 0 (B) 1 (C) 2 (D) 5 (E) infinitely many



Problem 21. An integer $n \geq 0$ is called *automorphic* if the decimal expansion of n^2 ends in the same digits as n (in order). For example, 0 is automorphic, since $0^2 = 0$, and 76 is automorphic, since $76^2 = 5776$. How many automorphic numbers are there between 0 and $10,000$, inclusive?

(A) 4 (B) 6 (C) 8 (D) 9 (E) 10

Problem 22. For $1 \le m < n$ consider the equation

$$1 + 2 + \cdots + m = (m+1) + (m+2) + \cdots + n\,.$$

There are two "small" solutions, namely $(m,n) = (2,3)$ and $(m,n) = (14,20)$. How many solutions are there with $20 < n \le 2012$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 23. Recall that $[x]$ denotes the greatest integer $\le x$. Determine

$$\left[\frac{1}{1^{2/3}} + \frac{1}{2^{2/3}} + \cdots + \frac{1}{1000^{2/3}}\right].$$

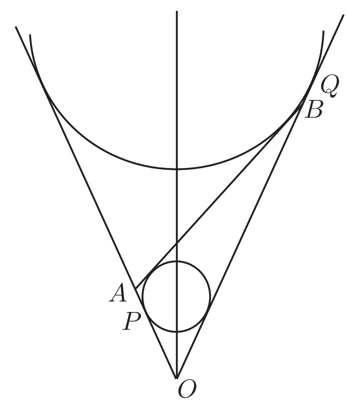
(A) 22 (B) 26 (C) 28 (D) 499 (E) None of the above

Problem 24. The Euler function $\phi(n)$ is defined to be the number of integers between 1 and n (inclusive) which do not share any common factor > 1 with n. For example, $\phi(10) = 4$, since $1, 3, 7$, and 9 is the full list of the numbers in $\{1, 2, 3, \ldots, 10\}$ without a common factor with 10. For how many integers $1 \le n \le 2012$ is $\phi(n) = n/3$?

(A) 4 (B) 30 (C) 60 (D) 335 (E) None of the above

Problem 25. Two circles, of radii 1 and 3, respectively, are inscribed in $\angle POQ$, as shown. They are also tangent to $\overline{AB}$, with P on $\overline{OA}$ and B on $\overline{OQ}$. If $OP = 3$, what is AB?

(A) 4 (B) 5 (C) $4\sqrt{3}$ (D) 6 (E) $6\sqrt{3}$



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