ALGEBRA QUALIFYING EXAM, SPRING 2025

Instructions: Complete all 8 problems. Each problem is worth 10 points. In multi-part problems, you may assume the result of any part (even if you have not been able to do it) in working on subsequent parts. Make sure to justify your answers.

(1) (a) Precisely state the third Sylow theorem that provides information on the number of Sylow p-subgroups for a finite group G.
(b) Let G be a group of order 380. Using the third Sylow theorem, give the possible number of Sylow 2-subgroups, Sylow 5-subgroups, and Sylow 19-subgroups.
(c) Using Sylow theory, prove that every group of order 380 is solvable.

(2) (a) Define precisely what it means for a finite extension field L over F to be a Galois extension.
(b) Construct a field extension L of $\mathbb{Q}$ such that $\mathrm{Gal}(L/\mathbb{Q})$ is isomorphic to S_3 (symmetric group on 3-letters). Provide clear justification for your answer.
(c) Show that there is an intermediate subfield K where $\mathbb{Q} \leq K \leq L$ such that $\mathrm{Gal}(L/K)$ is isomorphic to A_3.

(3) Let A be an $n \times n$ matrix over the real numbers $\mathbb{R}$. One can make $\mathbb{R}^n$ into a $\mathbb{R}[x]$-module by letting $f(x).v = f(A)(v)$ for $f(x) \in \mathbb{R}[x]$ and $v \in \mathbb{R}^n$. Assume that the module $\mathbb{R}^n$ has the following direct sum decomposition:

$$\mathbb{R}^n \cong \frac{\mathbb{R}[x]}{\langle (x+5)(x^2+4)^3 \rangle} \oplus \frac{\mathbb{R}[x]}{\langle (x-1)(x^2-1)(x^2+4)^4 \rangle} \oplus \frac{\mathbb{R}[x]}{\langle (x+2)(x^2+1)^2 \rangle}.$$

(a) Determine the elementary divisors and invariant factors of A.
(b) Determine the minimal polynomial of A.
(c) Determine the characteristic polynomial of A.

(4) Consider the following 4×4-matrix.

$$B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$$

(a) Determine the characteristic polynomial of B.
(b) Find a matrix in Jordan Canonical Form that is similar to B.

(5) Let $R = k[x]/\langle x^3 \rangle$ for a field k.
(a) Find all units in R.
(b) Show every ideal in R is principal.
(c) How many isomorphism classes of R modules are there which are 7-dimensional as a vector space over k?

(6) Let G be a finite p-group.
   (a) Show if X is a finite set with a G action, then $|X| = |X^G|$ modulo p.
   (b) Prove that G has non-trivial center.
   (c) Prove that G is solvable.

(7) Let $p(x) = x^7 + 6x^5 - 9x + 3$. Assume that $p(x)$ is irreducible over $\mathbb{Q}$. Let $L/\mathbb{Q}$ be a splitting field for $p(x)$. Viewing $G = \text{Gal}(L/\mathbb{Q})$ as a subgroup of S_7 by permuting the roots, show that G contains a 7-cycle.

   EXTRA CREDIT: For an additional 3 points, show that $p(x)$ is irreducible over $\mathbb{Q}$.

(8) Let $G \leq GL_n(k)$ be a finite subgroup of matrices over a field k. Here $GL_n(k)$ is the set of $n \times n$ invertible matrices over k.
   (a) Show that if $k = \mathbb{C}$ then every element of G is diagonalizable.
   (b) Give an example (not over $\mathbb{C}$) of a group $G \leq GL_n(k)$ such that every non-identity element has a minimal polynomial that is not square free.