Department of Mathematics
PRELIMINARY EXAMINATION

August 12, 2008

9–12 am

Work all of the following problems, justifying your answers. The problems are weighted evenly.

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{Z}$ denotes the set of all integers.

1. Discuss the truth of each of the following statements.
   a. For each integer x, there is an integer y so that $x + y$ is odd.
   b. There is an integer x so that $x + y$ is odd for each integer y.
   c. For each integer x there is an integer y so that the product xy is odd.
   d. There is an integer x so that the product xy is even for each integer y.

2. Let $F\colon \mathbb{R} \to \mathbb{R}$ be defined by $F(x) = \displaystyle\int_3^{\sin x} \sqrt{1 + t^4}\, dt$. Compute the derivative $F'(x)$ and carefully state the theorems needed to justify your answer.

3. For $n \in \mathbb{N}$, let $s_n = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \cdots + \dfrac{1}{n^2}$.
   a. Prove that $s_n \le 2 - \dfrac{1}{n}$ for all $n \in \mathbb{N}$.
   b. Prove that $\{s_n\}$ is a Cauchy sequence.

4. Let $A = \begin{bmatrix} -1 & 0 & 0 \\ -2 & 3 & -2 \\ -2 & 3 & -2 \end{bmatrix}$. Prove that A is diagonalizable and use this fact to compute A^{19}.

5. Suppose U and V are subspaces of a vector space X with $U \cap V = \{0\}$. Define

$$U + V = \{x \in X : x = u + v \text{ for some } u \in U \text{ and } v \in V\}.$$

   Suppose $\{u_1, \ldots, u_m\}$ is a basis for U and $\{v_1, \ldots, v_n\}$ is a basis for V. Prove that $\{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ is a basis for $U + V$.

6. Suppose $f:[0,1]\to\mathbb{R}$ is a continuous function, $f(x)\geq 0$ for all $x\in[0,1]$, and $f(1/2)>0$. Prove that $\displaystyle\int_0^1 f(x)\,dx>0$. Give an example to show that the result need not hold when f fails to be continuous.

7. Let $S\subset\mathbb{Z}$ be a nonempty subset that is closed under addition and additive inverse (i.e., if $a,b\in S$, then $a+b\in S$ and $-a\in S$). Prove that there is an integer k so that $S=\{nk : n\in\mathbb{Z}\}$. (You may use the division algorithm for $\mathbb{Z}$ but no fancy theorems.)

8. Give examples of the following. No proofs are required.
   a. a power series whose domain of convergence is the singleton set $\{5\}$
   b. functions $f,g:\mathbb{R}\to\mathbb{R}$, both of which are differentiable at 0 but whose composition $f\circ g$ is not differentiable at 0
   c. a function $f:\mathbb{R}^2\to\mathbb{R}$ that is continuous in each variable separately but is not continuous at $(0,0)$
   d. a sequence $\{a_n\}$ that converges to 0, where the partial sums of the series $\sum a_n$ are bounded and yet the series diverges
   e. a continuously differentiable vector field $F(x,y)=\big(M(x,y),N(x,y)\big)$ defined on $\mathbb{R}^2$ so that $\displaystyle\int_C M(x,y)\,dx+N(x,y)\,dy=4$ when C is the unit circle, oriented counterclockwise