Numerical Analysis Qualifying Exam, Spring, 2017

NAME______________________________ UGA Id. ____________________ Score ______________

Instruction: Do each problem using one page and show all your work. Put your name on all pages.

- [1]Suppose that a square matrix A is symmetric and positive definite. Show that when applying Gauss-Seidel iteration to solve $Ax = b$, the iteration converges.

- [2]Let A be a tridiagonal matrix. Write $A = QR$ with R being upper triangular matrix and Q orthonormal matrix. Show that RQ is also a tridiagonal matrix.

- [3]Let $f(x)$ be a continuous function over $x \in [0,1]^n$ with $n \geq 2$. Explain how to use the bisection method to find a zero inside $(0,1)^n$ if f changes sign over some vertices of the cube, say $f(0,0) > 0$ and $f(1,1) < 0$ for f defined over $[0,1] \times [0,1]$.

- [4]If using the following formula to compute an approximation of $f'(x)$:

  $$f'(x) \approx \frac{1}{12h}[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)],$$

  find the order of convergence as $h \to 0$.

- [5]Let $\triangle := \{\cdots \leq x_{-1} \leq x_0 \leq x_1 \leq \cdots\}$ be a nondecreasing knot sequence. Suppose that $x_{i+m} - x_i > 0$ for all i. Define the m^{th} order B-spline functions over $\triangle$ by

  $$B_i^m(x) = (x_{i+m} - x_i)[x_i, \cdots, x_{i+m}](\cdot - x)_+^{m-1}.$$

  Show that B-splines satisfy $B_i^m(x) \geq 0$ for all i and

  $$\sum_{i=k-m+1}^{k-1+j} B_i^m(x) = 1, \quad \forall x \in (x_k, x_{k+j})$$

  by using induction.

- [6]Derive the modified Euler's method:

  $$x(t+h) = x(t) + hf(t + h/2, x(t) + f(t, x(t))/2)$$

  by performing Richardson's extrapolation on Euler's method using step size h and $h/2$. Show that the truncated error is $O(h^2)$.