Numerical Analysis Qualifying Examination

Fall, 2006

Name__________________________

Instruction: Among following ten problems, please do 3 out of Problems 1-4, do 3 out of Problem 6-9, and do Problems 5 and 10. Please start each problem on a separate sheet of paper; write on only one side of the paper, and number each page. The time limit on this exam is three hours.

1. Suppose that a square matrix \( A = [a_{ij}]_{0 \leq i, j \leq n} \) is strictly diagonally dominant in the sense that \( a_{ii} \geq \sum_{j \neq i} |a_{ji}| \) for \( i = 1, \cdots, n \). Show that when applying Gaussian elimination procedure with partial pivoting to solve \( Ax = b \), there is no partial pivoting needed.

2. For any nonsingular matrix \( A \) of size \( n \times n \), denote \( \kappa(A) = \|A\|\|A^{-1}\| \) to be the condition number of \( A \), where \( \| \cdot \| \) is a subordinate (operator) norm.
   (1) Show that \( \kappa(A) \geq 1 \), \( \kappa(\alpha A) = \kappa(A) \) for any nonzero scalar \( \alpha \), and \( \kappa(AB) \leq \kappa(A)\kappa(B) \) for any matrix \( B \) of size \( n \times n \);
   (2) Further show the following
   \[
   \kappa(A) \geq \frac{\|A\|}{\|A - B\|}
   \]
   for any singular matrix \( B \) of size \( n \times n \).

3. Let \( A \) be an invertible matrix and \( \tilde{A} \) be a perturbation of \( A \) satisfying \( \|A^{-1}\| \|A - \tilde{A}\| < 1 \). Suppose that \( x \) and \( \tilde{x} \) are the exact solutions of \( Ax = b \) and \( \tilde{A}\tilde{x} = \tilde{b} \), respectively. Show that
   \[
   \frac{\|x - \tilde{x}\|}{\|x\|} \leq \frac{\kappa(A)}{1 - \kappa(A)\|A - \tilde{A}\|/\|A\|} \left[ \frac{\|A - \tilde{A}\|}{\|A\|} + \frac{\|b - \tilde{b}\|}{\|b\|} \right].
   \]

4. Let \( U, \Sigma, V \) be the singular value decomposition(SVD) of \( A \). Let
   \[
   A^+ = V \Sigma^+ U^T
   \]
   be the pseudo inverse of \( A \), where \( \Sigma^+ = \text{diag}\left(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \cdots, \frac{1}{\sigma_r}, 0, \cdots, 0\right) \) and \( r \) stands for the rank of \( A \). Show that
   \[
   AA^+A = A \text{ and } (A^+A)^T = A^+A.
   \]

5. Suppose that \( p \) is a root of multiplicity \( m > 1 \) of \( f(x) = 0 \). Show that the following modified Newton's method
   \[
   p_{n+1} = p_n - \frac{mf(p_n)}{f'(p_n)}
   \]
gives quadratic convergence.

[6] Show that the third order finite difference of \( f \) approximates \( f''' \) at the order \( O(h^2) \), i.e.,

\[
\frac{f(a_3) - 3f(a_2) + 3f(a_1) - f(a_0)}{h^3} - \frac{f'''((a_1 + a_2)/2)}{8} = \frac{h^2}{8} f^{(5)}(\xi)
\]

for some \( \xi \in [a_0, a_3] \), assuming \( a_i = a + i \times h, i = 0, 1, 2, 3 \).

[7] Suppose that \( f \in C^4[a, b] \). Prove the following error estimate for the Simpson rule:

\[
\int_a^b f(x)dx - \frac{(b - a)}{6} (f(a) + 4f(\frac{a + b}{2}) + f(b)) = -\frac{(b - a)^5}{2880} f^{(4)}(\xi)
\]

for some \( \xi \in [a, b] \).

[8] Suppose that \( f \in C[a, b] \). Let \( G_n(f) \) be the \( n^{th} \) Gaussian quadrature formula over interval \([a, b]\). That is, let \( \{x_i^{(n)}, i = 1, \ldots, n\} \) be the roots of orthonormal polynomials \( \phi_n \) of degree \( n \) for \( n \geq 1 \). The well-known Gaussian quadrature is defined by

\[
G_n(f) := \sum_{i=1}^{n} f(x_i^{(n)})a_i
\]

with \( a_i = \int_a^b \prod_{j=1}^{n} \frac{x - x_j^{(n)}}{(x_i^{(n)} - x_j^{(n)})} dx, i = 1, \ldots, n \). Show that

\[
G_n(f) \longrightarrow \int_a^b f(x)dx
\]

as \( n \to +\infty \).

[9] Let \( a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b \) be a partition of \([a, b]\). For \( f \in C^1[a, b] \), let \( S_f \) be the \( C^1 \) cubic interpolatory spline of \( f \), i.e.,

\[
S_f(x_i) = f(x_i), S'_f(x_i) = f'(x_i), i = 0, 1, \ldots, n + 1
\]

and \( S_f(x)|_{[x_i, x_{i+1}]} \) is a cubic polynomial, \( i = 0, \ldots, n \). Suppose that \( f \in C^2[a, b] \).

Show that

\[
\int_a^b \left| \frac{d^2}{dx^2} (f(x) - S_f(x)) \right|^2 dx \leq \int_a^b \left| \frac{d^2}{dx^2} f(x) \right|^2 dx.
\]

[10] Consider a single step method \( y_{k+1} = y_k + h\psi(x_k, y_k, h) \) for numerical solution of initial value problem of ODE \( y' = f(x, y) \). Suppose that \( \psi(x, y, h) \) is Lipschitz continuous with respect to \( y \) with Lipschitz constant \( L \). Suppose that the local truncation error of order \( m \), i.e., \( T_k(h) = \frac{y(x_{k+1}) - y(x_k)}{h} - \psi(x_k, y(x_k), h) = O(h^m) \). Show that numerical solution \( y_k \) approximates \( y(x_k) \) in the following sense

\[
|y(x_k) - y_k| \leq e^{(b-a) L} |y(x_0) - y_0| + e^{(b-a) L} \frac{1}{L} C h^m,
\]

for \( k = 1, \ldots, n \).