Numerical Analysis Qualifying Examination

Fall, 2006

Name____

Instruction: *Among following ten problems, please do 3 out of Problems 1–4, do 3 out of Problem 6–9, and do Problems 5 and 10. Please start each problem on a separate sheet of paper, write on only one side of the paper, and number each page. The time limit on this exam is three hours.*

[1] Suppose that a square matrix $A = [a_{ij}]_{0\le i,j\le n}$ is strictly diagonally dominant in the sense that $a_{ii} \ge \sum_{j\neq i} |a_{ji}$ for $i = 1, \cdots, n$. Show that when applying Gaussian elimination procedure with partial pivoting to solve $Ax = b$, there is no partial pivoting needed.

[2] For any nonsingular matrix A of size $n \times n$, denote $\kappa(A) = \|A\|\|A^{-1}\|$ to be the condition number of A, where $\|\cdot\|$ is a subordinate (operator) norm.

(1) Show that $\kappa(A) \ge 1$, $\kappa(\alpha A) = \kappa(A)$ for any nonzero scalar α, and $\kappa(AB) \le \kappa(A)\kappa(B)$ for any matrix B of size $n \times n$;

(2) Further show the following

$$\kappa(A) \ge \frac{\|A\|}{\|A - B\|}$$

for any singular matrix B of size $n \times n$.

[3] Let A be an invertible matrix and $\tilde{A}$ be a perturbation of A satisfying $\|A^{-1}\|\,\|A-\tilde{A}\| < 1$. Suppose that x and $\tilde{x}$ are the exact solutions of $Ax = b$ and $\tilde{A}\tilde{x} = \tilde{b}$, respectively. Show that

$$\frac{\|x-\tilde{x}\|}{\|x\|} \le \frac{\kappa(A)}{1-\kappa(A)\frac{\|A-\tilde{A}\|}{\|A\|}} \left[\frac{\|A-\tilde{A}\|}{\|A\|} + \frac{\|b-\tilde{b}\|}{\|b\|}\right].$$

[4] Let U, Σ, V be the singular value decomposition(SVD) of A. Let

$$A^+ = V\Sigma^+U^T$$

be the pseudo inverse of A, where $\Sigma^+ = \text{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \cdots, \frac{1}{\sigma_r}, 0, \cdots, 0)$ and r stands for the rank of A. Show that

$$AA^+A = A \text{ and } (A^+A)^T = A^+A.$$

[5] Suppose that p is a root of multiplicity $m > 1$ of $f(x) = 0$. Show that the following modified Newton's method

$$p_{n+1} = p_n - \frac{mf(p_n)}{f'(p_n)}$$

gives quadratic convergence.

[6] Show that the third order finite difference of f approximates f''' at the order $O(h^2)$, i.e.,

$$\frac{f(a_3) - 3f(a_2) + 3f(a_1) - f(a_0)}{h^3} - f'''((a_1 + a_2)/2) = \frac{h^2}{8} f^{(5)}(\xi)$$

for some $\xi \in [a_0, a_3]$, assuming $a_i = a + i * h, i = 0, 1, 2, 3$.

[7] Suppose that $f \in C^4[a, b]$. Prove the following error estimate for the Simpson rule:

$$\int_a^b f(x)dx - \frac{(b-a)}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b)) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi)$$

for some $\xi \in [a, b]$.

[8] Suppose that $f \in C[a, b]$. Let $G_n(f)$ be the n^{th} Gaussian quadrature formula over interval $[a, b]$. That is, let $\{x_i^{(n)}, i = 1, \ldots, n\}$ be the roots of orthonormal polynomials ϕ_n of degree n for $n \geq 1$. The well-known Gaussian quadrature is defined by

$$G_n(f) := \sum_{i=1}^{n} f(x_i^{(n)}) a_i$$

with $a_i = \int_a^b \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(x - x_j^{(n)})}{(x_i^{(n)} - x_j^{(n)})} dx, i = 1, \cdots, n$. Show that

$$G_n(f) \longrightarrow \int_a^b f(x)dx$$

as $n \longrightarrow +\infty$.

[9] Let $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$ be a partition of $[a, b]$. For $f \in C^1[a, b]$, let S_f be the C^1 cubic interpolatory spline of f, i.e.,

$$S_f(x_i) = f(x_i), S_f'(x_i) = f'(x_i), i = 0, 1, \cdots, n+1$$

and $S_f(x)|_{[x_i, x_{i+1}]}$ is a cubic polynomial, $i = 0, \cdots, n$. Suppose that $f \in C^2[a, b]$. Show that

$$\int_a^b |\frac{d^2}{dx^2}(f(x) - S_f(x))|^2 dx \leq \int_a^b |\frac{d^2}{dx^2} f(x)|^2 dx.$$

[10] Consider a single step method $y_{k+1} = y_k + h\psi(x_k, y_k, h)$ for numerical solution of initial value problem of ODE $y' = f(x, y)$. Suppose that $\psi(x, y, h)$ is Lipschitz continous with respect to y with Lipschitz constant L. Suppose that the local truncation error of order m, i.e., $T_k(h) = \frac{y(x_{k+1}) - y(x_k)}{h} - \psi(x_k, y(x_k), h) = O(h^m)$. Show that numerical solution y_k approximates $y(x_k)$ in the following sense

$$|y(x_k) - y_k| \leq e^{(b-a)L}|y(x_0) - y_0| + \frac{e^{(b-a)L} - 1}{L} Ch^m,$$

for $k = 1, \cdots, n$.