Problem 1. Both polynomials $P^{(1)}(x) = x^4 - 2x^3 + 2x^2 - 2x + 1$ and $P^{(2)}(x) = x^4 - x^3 + x - 1$ have a root $x^* = 1$. However, the errors between two consecutive Newton’s iterations $e_k^{(i)} = |x_{k+1}^{(i)} - x_k^{(i)}|, k \geq 0$ with $x_0^{(i)} = 1.2$ are shown in the following table for $P^{(i)}, i = 1, 2$:

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td>0.910 · 10⁻¹</td>
<td>0.517 · 10⁻¹</td>
<td>0.278 · 10⁻¹</td>
<td>0.145 · 10⁻¹</td>
<td>0.741 · 10⁻²</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>0.152</td>
<td>0.448 · 10⁻¹</td>
<td>0.329 · 10⁻²</td>
<td>0.163 · 10⁻⁴</td>
<td>0.398 · 10⁻⁹</td>
</tr>
</tbody>
</table>

The errors for $P^{(1)}$ decrease much slower.

a) (2 points) Explain the reason for this phenomenon.

b) (2 points) What can be done to accelerate the convergence for the first polynomial $P^{(1)}$?

c) (6 points) Justify your suggestion with a mathematical proof.

Problem 2.

a) (5 points) Define the Gaussian quadrature $G_n(f)$ carefully for a continuous function $f$.

b) (5 points) State the convergence theorem of the Gaussian quadratures $G_n(f)$ to the integral of $f$ and give a proof.

Problem 3. Suppose that $A \in \mathbb{R}^{n \times n}$ is nonsingular. Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ be two vectors. Under what condition is the following matrix

$$
\begin{pmatrix}
0, & \vec{u}^T \\
\vec{v}, & A
\end{pmatrix}
$$

invertible (4 points)? Find the explicit form for the inverse matrix (6 points).

Problem 4. Let

$$U_n(x) := \frac{\sin((n + 1) \arccos x)}{\sqrt{1 - x^2}}, \quad n = 0, 1, 2, \ldots.$$

a) (5 points) Prove that these functions are algebraic polynomials of degree $n$. 
b)(5 points) For integers \( n \neq m \), prove
\[
\int_{-1}^{1} \sqrt{1 - x^2} U_n(x) U_m(x) dx = 0.
\]

**Problem 5.** Fix \( n \geq 1 \). Let \( B_n \) be a polynomial B-spline of degree \( n \) with integer nodes and support on \([0, n+1]\). Prove that \( \{B_n(x - i)\}_{i=-n}^{N-1} \) are linearly independent on the interval \([0, N]\), \( N \geq 1 \).

**Problem 6.** An \((n+1)\)-dimensional linear subspace \( H \) of \( C[a,b] \) is called a Haar subspace if each non-zero function in \( H \) has at most \( n \) roots.

Show that the linear span \( H \) of functions \( \{1, x, x^2, \ldots, x^{n-1}, f(x)\} \) is a Haar subspace of \( C([a, b]) \) if the \( n^{th} \) derivative \( f^{(n)}(x) \) of \( f \) is strictly positive on \([a, b]\).

**Problem 7.** Suppose that the matrix norm \( \| \cdot \| \) is subordinate. Let \( S \) be a non-singular square matrix. Prove or disprove that \( \|A\|_* := \|SAS^{-1}\| \) is also a subordinate norm.

**Problem 8.** Suppose that a square matrix \( A \) is strictly diagonally dominant. Show that the Gauss-Seidel iteration for the linear system \( Ax = b \) converges.

**Problem 9.** Consider a single step method \( y_{k+1} = y_k + h\psi(x_k, y_k, h) \) for numerical solution of initial value problem of ODE \( y' = f(x, y) \). Suppose that \( \psi(x, y, h) \) is Lipschitz continuous with respect to \( y \) with Lipschitz constant \( L \). Suppose that the local truncation error of order \( m \), i.e., \( T_k(h) = \frac{y(x_{k+1}) - y(x_k)}{h} - \psi(x_k, y(x_k), h) = O(h^m) \). Show that numerical solution \( y_k \) approximates \( y(x_k) \) in the following sense
\[
|y(x_k) - y_k| \leq e^{(b-a)L}|y(x_0) - y_0| + \frac{e^{(b-a)L} - 1}{L} Ch^m,
\]
for \( k = 1, \ldots, n \).

**Problem 10.** Consider a linear least squares problem:
\[
\min \|Ax - b\|^2,
\]
where \( A \) is a matrix of size \( m \times n \) with \( m > n \) and \( b \) of size \( m \times 1 \). (a) Use the SVD to describe how to solve the least square problem (5 points), and (b) explain why your method works (5 points).