Top

PRELIMINARY EXAMINATION IN TOPOLOGY

MAY 8, 1992

Directions: Do all the problems. Problems #1-6 are each worth 10 points; #7 and #8 are each worth 20 points.

1. Give a self-contained proof of the following:

   Let X be a compact metric space. Given any open covering $\mathcal{U}$ of X, prove that there is a real number $\epsilon > 0$ so that for each $x \in X$, there is $U \in \mathcal{U}$ such that $B(x,\epsilon) \subset U$.

2. a. Prove that if Y is a retract of the Hausdorff space Z, then Y is a closed subspace of Z.

   b. Let J be an arbitrary set; endow $Z = \prod_{j \in J} \mathbb{R}$ with the product topology. Prove that if Y is a retract of Z, then for every normal topological space X, closed subspace $A \subset X$ and continuous function $f : A \to Y$, there exists a continuous extension $\bar{f} : X \to Y$.

3. Let X be the set of real numbers endowed with the topology generated by basis elements $[a,b)$, $a, b \in \mathbb{R}$, $a < b$. Let $\mathbb{R}$ denote the set of real numbers endowed with the standard topology.

   a. Classify all continuous functions $f : \mathbb{R} \to X$.

   b. Classify all continuous functions $f : X \to \mathbb{R}$.

4. Prove or give a counterexample in each case: If X is a contractible space, then

   a. X is simply connected.

   b. X is locally simply connected at each point $x \in X$.

5. Let $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, and let $S^{n-1} = \partial D^n = \{x \in \mathbb{R}^n : |x| = 1\}$. Suppose $f : D^n \to \mathbb{R}^n$ is continuous and satisfies $|f(x) - x| < 1$ for all $x \in S^{n-1}$. Prove that $0 \in f(D^n)$.

6. View the torus T as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$, so we have the obvious covering map $\pi : \mathbb{R}^2 \to T$. The matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ defines a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$.

   a. Prove briefly that this linear map induces a continuous map $f : T \to T$.

   b. Prove or disprove: f is homotopic to the identity map.

—OVER—

Top

7. Let S^2, $S^{2\prime}$ be two copies of the two-sphere, and let $p, q \in S^2$, $p', q' \in S^{2\prime}$ be pairs of points in the respective copies. Define

$$X = S^2 \cup S^{2\prime} / \left(p \sim p',\ q \sim q'\right).$$

a. Give the universal covering space of X.

b. Using Van Kampen's Theorem, compute $\pi_1(X)$ and relate your answer to your answer to a.

c. Compute $H_*(X, \mathbb{Z})$ by any method you desire. Give details.

8. a. Define the Lefschetz number $L(f)$ of a continuous map $f : X \to X$.

b. Let $f : \mathbb{RP}^2 \to \mathbb{RP}^2$ be continuous. Give (with proof) necessary and sufficient conditions for f to have a fixed point.

c. Let K be a simplicial complex. State and sketch a proof of the Lefschetz fixed point theorem for a continuous map $f : |K| \to |K|$.