Departmental Syllabus for MATH 2270: Calculus III for Science and Mathematics

Course description (from the UGA Bulletin): Calculus of functions of two and three variables: Parametric curves and applications to planetary motion. Derivatives, the gradient, Lagrange multipliers. Multiple integration, area, volume, and physical applications, polar, cylindrical, and spherical coordinates. Line and surface integrals, Green's, Stokes's, and Divergence theorems, with applications to physics.

Textbook: Currently, most sections of MATH 2270 use Volume 3 of *OpenStax Calculus* (freely available at https://openstax.org/details/books/calculus-volume-3), with the body of the course corresponding to Chapters 3-6. *University Calculus* by Hass, Heil, Weir, Bogacki, and Thomas (currently on its fourth edition, ISBN 0134995546) is also an option. References to sections in both books below are current as of April 2023.
For online homework, WebAssign (paid) offers assignments based on OpenStax, and there are also many problems on all Calculus III subjects in WeBWorK (free), including ones connected to sections in a previous edition of *University Calculus*.

Outline: MATH 2270 begins with a quick review of the material on vectors that was covered at the end of 2260. Note that students who took Calculus II somewhere other than UGA may be less familiar with this subject, as many Calculus II courses cover parametric curves and polar coordinates instead of vectors. This review is followed by units on vector-valued functions, partial differentiation, multiple integrals, and finally the two- and three-dimensional versions of the fundamental theorem of calculus that synthesize most of the rest of the course. Suggested durations below are based on a four class-period-per-week schedule, with six days left for review and tests.

Here are the topics covered in the course, with associated recommended learning outcomes:

0. **Review of vectors (OpenStax Chapter 2; HHWBT Chapter 11; 4 days)**
   • Compute the dot and cross products of two vectors.
   • Given two vectors in space, find the angle between them, and find a third vector orthogonal to both.
   • Find an equation for a plane given a point on it and a normal vector to it, or given three points on it.

1. **Vector-valued functions and motion in space (4 days)**
   a) Vector-valued functions and parametrized curves (OS 3.1; HHWBT start of 12.1; also include some discussion of ideas/examples of parametrizations from OS 1.1 or HHWBT 10.1; 1.5 days)
   • Find parametrizations for circles and other simple curves.
   • Modify a parametrization of a given curve to correspond to changes to the curve, such

as scalings, translations, or orientation-reversal.

b) Calculus of vector-valued functions (OS 3.2; HHWBT end of 12.1 and start of 12.2; 1 day)
- Given a parametrized curve and a point on it, find the unit tangent vector at that point.
- Find the integral of a vector-valued function.

c) Arc length (OS start of 3.3; HHWBT 12.3; 0.5 day)
- Given a parametrization of a curve, find its arc length.

d) Motion in space (OS 3.4; HHWBT end of 12.2; 1 day)
- Determine the trajectory of a particle given its acceleration and its initial position and velocity.
- Connect relationships between the velocity and acceleration vectors of an object to the motion of the object (*e.g.*, by the dot product rule, for motion on a sphere **v** and **a** will be perpendicular).

2. **Derivatives of functions of several variables (15 days)**

a) Introduction to functions of several variables (OS 4.1 and 4.2; HHWBT 13.1 and 13.2; 2 days)
- Sketch level curves for functions of two variables, and level surfaces for simple functions of three variables.
- Identify the different kinds of surfaces that can arise as graphs of quadratic functions of two variables.
- Show that certain two-variable limits do not exist by comparing limits along different paths of approach.

b) Partial derivatives (OS 4.3; HHWBT 13.3; 2 days)
- Compute and interpret first partial derivatives of functions of any number of variables.
- Compute higher-order partial derivatives of functions of any number of variables.

c) Tangent planes and linear approximation (OS 4.4; HHWBT 13.6; 2.5 days)
- Find an equation for the tangent plane to the graph of a function at a prescribed point.
- Based on information about a multivariable function and its partial derivatives at a single point, use linear approximation to estimate the value of the function at nearby points.

d) The chain rule (OS 4.5; HHWBT 13.4; 2 days)
- State and apply the chain rule for compositions of functions of one, two, and/or three variables.

• Perform implicit differentiation of a function of two or more variables.

e) Directional derivatives and the gradient vector (OS 4.6; HHWBT 13.5; 2 days)
• Find and correctly interpret the directional derivative of a function at a point in a given direction.
• Use the gradient vector to determine the direction of most rapid increase of a function at a point.
• Use the gradient vector to find the tangent line at a point to a level curve, or the tangent plane at a point to a level surface.

f) Multivariable maximum and minimum problems (OS 4.7; HHWBT 13.7; 2.5 days)
• Find the critical points of a function of two or three variables.
• Use the second derivative test to classify the critical points of a function of two variables as local minima, local maxima, or saddle points.
• Find the absolute extrema of a function on a bounded region by comparing critical values and boundary values.

g) Lagrange multipliers (OS 4.8; HHWBT 13.8; 2 days)
• Use the method of Lagrange multipliers to optimize a multivariable function subject to a single constraint.

3. **Multiple integrals (13 days)**
a) Double integrals over rectangles (OS 5.1; HHWBT 14.1; 2 days)
• Express a double integral over a rectangle as an iterated integral using either order of integration, and evaluate the integral.
• Use double integrals to find volumes between surfaces, and average values of functions over regions.

b) Double integrals over general regions (OS 5.2; HHWBT 14.2 and 14.3; 3 days)
• Set up and evaluate double integrals over the region between two graphs (of functions of *x* or of *y*), or over unions of such regions.
• Simplify the calculation of an iterated integral by changing the order of integration.
• Find areas of regions and average values of functions over regions by double integration.

c) Double integrals in polar coordinates (OS 1.3 and 5.3; HHWBT 10.3 and 14.4; 2 days)
• Convert between the rectangular (Cartesian) and polar coordinate representations of a point.
• Identify regions that can be expressed more simply in terms of polar coordinates than in terms of rectangular coordinates.
• Set up and evaluate integrals in polar coordinates.

d) Triple integrals in rectangular coordinates (OS 5.4; HHWBT 14.5; 1.5 days)
• Set up and evaluate triple integrals, both over rectangular boxes and over other regions.
• Use triple integrals to find average values.

e) Triple integrals in cylindrical and spherical coordinates (OS 5.5; HHWBT 14.7; 3 days)
• Convert between the rectangular and cylindrical coordinate representations of a point.
• Set up and evaluate integrals in cylindrical coordinates.
• Convert between the rectangular and spherical coordinate representations of a point.
• Set up and evaluate integrals in spherical coordinates.
• Given a triple integral problem, decide which of rectangular, cylindrical, and/or spherical coordinates are well-suited to problem.

f) Moments and centers of mass (OS 5.6; HHWBT 14.6; 1.5 days)
• Use a double or triple integral to compute the mass and the center of mass of a two- or three-dimensional object, possibly appealing to symmetry principles to simplify the calculation.
• Find the moment of inertia of a solid object around an axis.

4. **Vector calculus (18 days)**

a) Vector fields (OS 6.1; HHWBT start of 15.2; 1.5 days)
• Visualize vector fields in the plane, by sketching simple ones, or by matching given sketches of vector fields to formulas.
• Recognize when a vector field is the gradient of a given function, and when the cross-partial criterion implies that a vector field cannot be the gradient of any function.

b) Line integrals (OS 6.2; HHWBT 15.1 and end of 15.2; 3 days)
• Find line integrals of scalar functions along curves.
• Find line integrals of vector fields along oriented curves
• Use line integrals to calculate work, flux, or circulation.

c) Conservative vector fields (OS 6.3; HHWBT 15.3; 2.5 days)
• Use the fundamental theorem for line integrals to compute line integrals of conservative vector fields.
• Use the fundamental theorem for line integrals, in the form of the path-independence criterion, to show that some vector fields are not conservative.
• Find potential functions for conservative vector fields.

d) Green's theorem (OS 6.4; HHWBT 15.4; 2.5 days)

• Use Green's theorem, in either the circulation form or the flux form, to evaluate line integrals around closed curves.
• Use Green's theorem to find areas of regions enclosed by curves (by integrating *xdy* or *(xdy-ydx)/*2).

e) Divergence and curl (OS 6.5; HHWBT starts of 15.7 and 15.8; 1 day)
• Find and interpret the divergence of a vector field in three dimensions.
• Find and interpret the curl of a vector field in three dimensions.

f) Surface integrals (OS 6.6; HHWBT 15.6; 3.5 days)
• Find parametrizations for surfaces formed by parts of cones, spheres, and cylinders.
• Find areas of parametrized surfaces, and integrals of scalar functions over such surfaces.
• Set up and evaluate integrals for the flux of a vector field through an oriented surface, distinguishing between the two possible orientations of the surface.

g) Stokes' theorem (OS 6.7; HHWBT 15.7; 2 days)
• Given a vector field and an oriented surface, identify the line integral and the surface integral that are equated by Stokes' theorem.
• Use Stokes' theorem to compute a line integral by replacing it by a surface integral.
• Use Stokes' theorem to compute a surface integral by replacing it by a line integral.

h) The divergence theorem (OS 6.8; HHWBT 15.8; 2 days)
• Given a vector field and a region bounded by a closed surface, identify the surface integral and the triple integral that are equated by the divergence theorem.
• Use the divergence theorem to compute a surface integral by replacing it by a triple integral.
• Use the divergence theorem to compute a triple integral by replacing it by a surface integral.

(approved by the Curriculum Committee, April 2023)