Summary of 'A Simple Proof of the Restricted Isometry Property for Random Matrices'

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We will be restricting our attention to section 5 of the paper. A few notes before we begin:

- Φ will always be an $n \times N$ matrix whose elements are iid draws from a subgaussian distribution.
- For some such Φ, with $\epsilon \in (0,1)$, it is known that for any $x \in \mathbb{R}^N$ there exists a $c_0(\epsilon)$ such that:

$$\Pr(|\, \|\Phi x\|^2_{l_2^n} - \|x\|^2_{l_2^N} \,| \geq \epsilon \|x\|^2_{l_2^N}) \leq 2e^{-nc_0(\epsilon)} \tag{1}$$

- Given $T \subseteq \{1, 2, ..., N\}$, we denote by Φ_T the matrix obtained by selecting only the columns of Φ indexed by the elements of T. Note here that Φ_T is an $n \times \#(T)$ matrix.
- Given $T \subseteq \{1, 2, ..., N\}$ and $x \in \mathbb{R}^N$, we denote by x_T the vector in $\mathbb{R}^{\#(T)}$ whose elements are from x at the indices provided in T.

1 Introduction and Strategy

Firstly, Φ satisfies the Restricted Isometry Property of order k if $\exists \delta_k \in (0,1)$ such that

$$(1-\delta_k) \|x_T\|^2_{l_2^N} \leq \|\Phi_T x_T\|^2_{l_2^n} \leq (1+\delta_k) \|x_T\|^2_{l_2^N} \tag{2}$$

for all T with $\#(T) \leq k$.

Loosely speaking, our goal is to show that Φ satisfies the Restricted Isometry Property with high probability. Our strategy towards showing this is as follows:

- From R^N, we reduce our attention to a k-dimensional subspace X_T.
- Using elementary results from covering numbers, we can find a finite set of elements Q_T ($\#Q_T \leq (\frac{12}{\delta})^k$) such that for any $x \in X_T$, there's a q less than $\frac{\delta}{4}$ away.
- Using the concentration of measure from (1), we establish an upper bound for the probability of Restricted Isometry Property failing in any of our k-dimensional subspaces.
- Using union bounds to combine probabilities, we ultimately find an upper bound on the probability of our Φ failing the RIP given any k.

2 Section 2

Fix a $k \leq N$. For a given

$$T := \{i_1, i_2, ..., i_t : t \leq k\} \subseteq \{1, 2, ..., N\}$$

define:

$$X_T := \{[x_1, x_2, ..., x_N] : x_i = 0 \text{ for } i \notin T\}$$

or, the set of all vectors in $\mathbb{R}^N$ whose entries are 0 if their index is not in T. This is clearly a k-dimensional linear subspace of $\mathbb{R}^N$.

Lemma 2.1. *Given any set of indices T where $\#(T) = k < n$ and $\delta \in (0,1)$:*

$$(1-\delta) \|x\|_{l_2^N} \leq \|\Phi x\|_{l_2^n} \leq (1+\delta) \|x\|_{l_2^N} \tag{3}$$

for any $x \in X_T$ with probability

$$\geq 1 - 2(\frac{12}{\delta})^k e^{-c_0(\frac{\delta}{2})n} \tag{4}$$

Proof. Because Φ is linear, we can reduce our attention to only $x \in X_T$ such that $\|x\|_{l_2^N} \leq 1$. It is known that we can select a set $Q_T \subseteq X_T$ of points such that $\|q\|_{l_2^N} \leq 1$ for all $q \in Q_T$ and for any $x \in X_T$ where $\|x\|_{l_2^N} = 1$, $\exists q \in Q_T$ such that

$$x \in B_{\frac{\delta}{4}}(q)$$

i.e. given any $x \in X_T$ with norm 1, there's a $q \in Q_T$ "close by".

It is further known that we can choose this set such that $\#(Q_T) \leq (\frac{12}{\delta})^k$.

We know that for a given $q \in Q_T$, (1) holds with $\epsilon = \frac{\delta}{2}$ i.e.

$$\Pr(|\,\|\Phi x\|_{l_2^n}^2 - \|x\|_{l_2^N}^2\,| \geq (\frac{\delta}{2})\,\|x\|_{l_2^N}^2) \leq 2e^{-nc_0(\frac{\delta}{2})}$$

and hence, applying a union bound to all $q \in Q_T$, of which there are no more than $(\frac{12}{\delta})^k$:

$$\Pr(\exists q \in Q_T,\ |\,\|\Phi q\|_{l_2^n}^2 - \|q\|_{l_2^N}^2\,| \geq (\frac{\delta}{2})\,\|q\|_{l_2^N}^2) \leq 2e^{-nc_0(\frac{\delta}{2})}(\frac{12}{\delta})^k$$

Via complements:

$$(1 - \frac{\delta}{2})\,\|q\|_{l_2^N}^2 \leq \|\Phi q\|_{l_2^n}^2 \leq (1 + \frac{\delta}{2})\,\|q\|_{l_2^N}^2 \qquad (5)$$

with probability

$$\geq 1 - 2(\frac{12}{\delta})^k e^{-c_0(\frac{\delta}{2})n}$$

which is the precisely the bound (4) specified in our lemma. Now we will use the inequalities found in (5) to expand this bound to all $x \in X_T$.

From (5) we can see that:

$$(1 - \frac{\delta}{2})\,\|q\|_{l_2^N} \leq \|\Phi q\|_{l_2^n} \leq (1 + \frac{\delta}{2})\,\|q\|_{l_2^N}$$

We define A to be the smallest number with the property that

$$\|\Phi x\|_{l_2^n} \leq (1 + A)\,\|x\|_{l_2^N}$$

for all $x \in X_T$. Picking a $q \in Q_T$ such that $x \in B_{\frac{\delta}{4}}(q)$ and using the linearity of Φ:

$$\begin{aligned} \|\Phi x\|_{l_2^n} &= \|\Phi(q + x - q)\|_{l_n^2} \\ &= \|\Phi(q) + \Phi(x - q)\|_{l_2^n} \end{aligned}$$

And now by the triangle inequality we see:

$$\|\Phi(q) + \Phi(x - q)\|_{l_2^n} \leq \|\Phi(q)\|_{l_2^n} + \|\Phi(x - q)\|_{l_2^n}$$

And now, since $x - q \in X_T$, and since $\|q\|_{l_2^N} \leq 1$ we actually have that

$$\|\Phi(q)\|_{l_2^n} + \|\Phi(x - q)\|_{l_2^n} \leq (1 + \frac{\delta}{2}) + (1 + A)\frac{\delta}{4}$$

Now, since $\|x\|_{l_2^N} \leq 1$ we have that

$$\|\Phi x\|_{l_2^n} \leq (1 + (\frac{\delta}{2} + (1 + A)\frac{\delta}{4}))\,\|\Phi x\|_{l_2^n}$$

and hence, since A is the smallest real such that $\|\Phi x\|_{\ell_2^n} \leq (1 + A)\,\|x\|_{\ell_2^N}$

$$\begin{aligned} A &\leq \frac{\delta}{2} + (1 + A)\frac{\delta}{4} \\ &= \frac{2\delta}{4} + \frac{\delta}{4} + A\frac{\delta}{4} \\ &\iff A(1 - \frac{\delta}{4}) \leq \frac{3\delta}{4} \\ &\iff A \leq \frac{3\delta}{4(1 - \frac{\delta}{4})} = \frac{3\delta}{4 - \delta} \leq \frac{3\delta}{3} = \delta \end{aligned}$$

And so we see that $A \leq \delta$ which implies second inequality of (3).

To see the first inequality of (3), notice that

$$\begin{aligned} \|\Phi x\|_{l_2^n} &\geq \|\Phi(q)\|_{l_2^n} + \|\Phi(x - q)\|_{l_2^n} \\ &\geq (1 - \frac{\delta}{2}) - (1 + \delta)\frac{\delta}{4} \\ &\geq 1 - \delta \end{aligned}$$

□

We will now apply this lemma to show that Φ satisfies the Restricted Isometry Property of order k with high probability.

Theorem 2.2. *For a given $\delta \in (0, 1)$, $\exists c_1, c_2 \in \mathbb{R}$ such that with a probability greater than $1 - 2e^{-c_2 n}$, Φ will satisfy the Restricted Isometry Property of order $k \leq \frac{c_1 n}{\log(\frac{N}{k})}$.*

Proof. We know that there are only $\binom{N}{k} \leq (\frac{eN}{k})^k$ possibilities for the k-dimensional subspace discussed in Lemma 2.1. So for any $x \in \mathbb{R}^N$ we see that by using a union bound to combine probabilities:

$$(1-\delta)\left\|x\right\|_{l_2^N} \leq \left\|\Phi x\right\|_{l_2^n} \leq (1+\delta)\left\|x\right\|_{l_2^N}$$

fails to hold with a probability of no more than

$$2(\frac{eN}{k})^k(\frac{12}{\delta})^k e^{-c_0(\frac{\delta}{2})n} = 2e^{-c_0(\frac{\delta}{2})n+k[\log(\frac{eN}{k})+\log(\frac{12}{\delta})]}$$

Thus we see by picking

$$0 < c_2 \leq c_0(\delta) - c_1[1+(1+\frac{\log(\frac{12}{\delta}))}{\log(\frac{N}{k})})]$$

and since $k \leq \frac{c_1 n}{\log(\frac{N}{k})}$, we have that

$$\begin{aligned}
& -c_0(\frac{\delta}{2})n + k[\log(\frac{eN}{k}) + \log(\frac{12}{\delta})] \\
& \leq -c_0(\frac{\delta}{2})n + \frac{c_1 n}{\log(\frac{N}{k})}[\log(\frac{eN}{k}) + \log(\frac{12}{\delta})] \\
& = -n[c_0(\frac{\delta}{2}) - c_1(\frac{1}{\log(\frac{N}{k})})[\log(e) + \log(\frac{N}{k}) + \log(\frac{12}{\delta})]] \\
& = -n[c_0(\frac{\delta}{2}) - c_1[\frac{\log(e)}{\log(\frac{N}{k})} + 1 + \frac{\log(\frac{12}{\delta})}{\log(\frac{N}{k})}]] \\
& \leq -n[c_0(\frac{\delta}{2}) - c_1[1+(1+\frac{\log(\frac{12}{\delta})}{\log(\frac{N}{k})})]] \\
& \leq -nc_2
\end{aligned}$$

And so, we have shown $\exists c_1, c_2 > 0$ such that, so long as $k \leq c_1(\frac{n}{\log\frac{N}{k}})$, then our randomly generated Φ fails to satisfy the Restricted Isometry Property of order k with probability less than $2e^{-c_2 n}$. Hence by complements, Φ will satisfy the Restricted Isometry Property of order k with probability greater than $1-2e^{-c_2 n}$. □