Carl F. Kossack Calculus Prize Examination
April 6, 2024

Name: _____

Instructions

- This test has eight problems, and you have two hours to complete it.
- Fill out your answers in the blank space provided.
  You may use the back sides of pages. *There is a scratch page at the end.*
- **No aid of any kind** is allowed. Calculators are not allowed.
  Unsimplified answers are accepted unless the problem requests a specific answer format.
- Show your work, and give clear reasoning.

Good luck! The questions start on the next page.

Problem Number	Points Possible	Points Made
1	10	
2	10	
3	13	
4	15	
5	15	
6	12	
7	10	
8	15	
Total:	100	

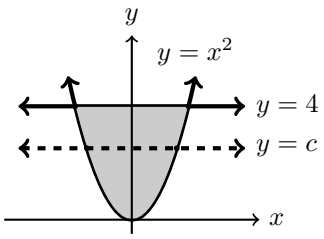
Carl F. Kossack Prize Exam 2024 Committee:
Michael Klipper (chair), Phil Bergonio, Matthew Just, Gordana Matic, Shuzhou Wang

1. [10 pts] Determine the following limit.

$$\lim_{x \to -\infty} (\sqrt{x^2 + x} - \sqrt{x^2 - x})$$

Points earned: _____
out of a possible 10 points

2. [10 pts] Consider the region enclosed by $y = x^2$ and $y = 4$, pictured below. Determine the value of c so that $y = c$ divides the region into two parts of the same area.



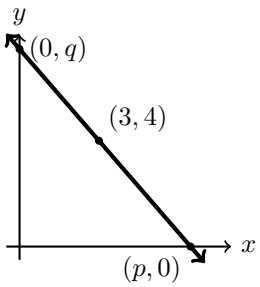
Points earned: _____
out of a possible 10 points

3. [13 pts] Compute the following integral, showing all work.

$$\int_0^1 \frac{dx}{(x^2+1)^3}$$

Points earned: _____
out of a possible 13 points

4. [15 pts] A straight line passes through the point $(3, 4)$, intersecting the x-axis at $x = p$ and the y-axis at $y = q$ (with $p, q > 0$). Determine the minimum possible value of $p + q$. Justify your answer.



Points earned: _____
out of a possible 15 points

5. [15 pts] Let $g(x)$ be a **polynomial of degree** 4 with the following properties:

   - The graph of $g(x)$ has critical points at $x = 0$, $x = -3$, and $x = 4$.
   - At $x = 1$, the tangent line to $g(x)$ is $y = 6x + 25$.

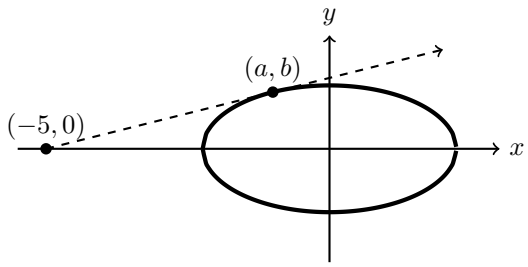
   Determine a possible formula for $g(x)$, with justification.

   (**HINT:** First write $g'(x)$ in factored form with an unknown leading coefficient.)

Points earned: _____
out of a possible 15 points

6. [12 pts] A spotlight located at the point $(-5, 0)$ shines **upward** at an ellipse $x^2 + 4y^2 = 5$. The light beam passes tangent to the ellipse through a point (a, b). Determine that point.

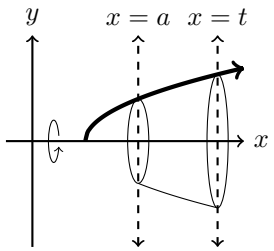
   (**HINT:** Find the slope of the line in two different ways, one of which uses implicit differentiation.)



Points earned: _____
out of a possible 12 points

7. [10 pts] Consider the region enclosed by some positive continuous function $y = f(x)$ and the x-axis between the lines $x = a$ and $x = t$ (where a is constant and $t > a$). When this region is rotated around the x-axis, the volume of revolution is $V(t) = t^2 - at$. Determine a formula for $f(x)$, with justification. (The answer may use a in it.)

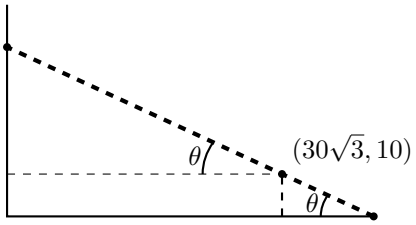
   (**HINT:** Which part of the Fundamental Theorem can be used here?)



Points earned: _____
out of a possible 10 points

8. [15 pts] In the following picture, imagine an infinitely-tall wall. A lamppost is 10 feet tall and is located $30\sqrt{3}$ feet to the right of the wall. We want to place a ladder (drawn in thick dashes along the hypotenuse) with its base on the ground with its top on the wall, so that the ladder passes over the top of the lamp. Let θ be the angle of elevation of the ladder. If $L(\theta)$ determines the ladder length, determine its critical point. (You may assume this produces the absolute minimum length.)

   (**HINT:** After setting $L'(\theta) = 0$, rearrange the equation to get $\tan^3(\theta)$ equaling a constant.)



Points earned: _____
out of a possible 15 points

Extra space for work. **Do not detach this page.**

If you want us to consider the work on this page,
please indicate which problem(s) the work goes with!