Carl F. Kossack Calculus Prize Examination

March 29, 2025

SOLUTIONS (by Michael Klipper unless otherwise specified)

Name: ______________________________

Instructions

- This test has eight problems, and you have two hours to complete it.
- Fill out your answers in the blank space provided.
  You may use the back sides of pages. *There is a scratch page at the end.*
- **No aid of any kind** is allowed. Calculators are not allowed.
  Unsimplified answers are accepted unless the problem requests a specific answer format.
- Show your work, and give clear reasoning.

Good luck! The questions start on the next page.

Problem Number	Points Possible	Points Made
1	10	
2	12	
3	13	
4	10	
5	12	
6	13	
7	15	
8	15	
Total:	100	

Carl F. Kossack Prize Exam 2025 Committee:
Michael Klipper (chair), Matthew Just, Nur Saglam, Shuzhou Wang

1. [10 pts] Find a cubic polynomial of the form $p(x) = x^3 + ax^2 + bx + c$ which has a local extremum at $(3, 2)$ and a point of inflection at $x = 1$.

Solution:

There are three clues, each of which gives us one unknown.

- Since there's a point of inflection at $x = 1$, it means $p''(x)$ changes sign at $x = 1$. Note $p'(x) = 3x^2 + 2ax + b$ and $p''(x) = 6x + 2a$, so $p''(1) = 0$.[1] Therefore, $6 + 2a = 0$, hence $2a = -6$ and $\boxed{a = -3}$.

- There's a local extremum at $x = 3$, so $p'(x)$ changes sign at $x = 3$, and hence $p'(3) = 0$. Using $p'(x) = 3x^2 + 2(-3)x + b$, replacing $a = -3$ as found in the last step, we get $3(3)^2 - 6(3) + b = 0 = 27 - 18 + b = 9 + b$. Therefore, $\boxed{b = -9}$.

- Finally, since $(3, 2)$ is on the curve, it means $p(3) = 2$. Plug in our a and b values, along with $p(3) = 2$, to get $2 = (3)^3 + (-3)(3)^2 + (-9)(3) + c = 27 - 27 - 27 + c$, so $-27 + c = 2$ and hence $\boxed{c = 29}$.

[1] Since $p''(x)$ is a polynomial, it is continuous, and the only way a continuous function can change sign is to cross over an x-intercept. (If $p''(x)$ were discontinuous, it could be possible for $p''(x)$ to have some jump at $x = 1$, and then $p''(1)$ DNE.) This is a minor technicality though.

Points earned: ______
out of a possible 10 points

2. [12 pts] Determine the derivative of the function $f(x) = (x-2025)^{(x+1)^x}$.
(Here, $(x+1)^x$ is the exponent on top of $(x-2025)$.)

Solution:

Because each base and exponent is not constant, the exponential derivative law $(b^x)' = b^x \ln(b)$ is **not applicable to this problem**. Instead, the standard tactic is to rewrite every exponential to use base e, i.e. $g(x) = e^{\ln(g(x))}$ aka $\exp(g(x))$. This would produce

$$f(x) = (x-2025)^{(x+1)^x} = \exp\left(\ln(x-2025)\cdot(x+1)^x\right) = \exp\left(\ln(x-2025)\cdot \exp(\ln(x+1)\cdot x)\right)$$

From here, we use Chain and Product Rules.

For a cleaner way to write this work, use logarithmic differentiation![2]

$$\ln(f(x)) = \ln\left[(x-2025)^{(x+1)^x}\right] = (x+1)^x \cdot \ln[x-2025]$$

Since this has another exponential with nonconstant base and exponent, take **another** logarithm! We use a couple different log identities to simplify this:

$$\begin{aligned} \ln(\ln(f(x))) &= \ln\left[(x+1)^x \cdot \ln(x-2025)\right] = \ln\left[(x+1)^x\right] + \ln\left[\ln(x-2025)\right] \\ &= x \cdot \ln(x+1) + \ln(\ln(x-2025)) \end{aligned}$$

Now differentiate both sides with respect to x:

$$\frac{1}{\ln(f(x))} \cdot \frac{1}{f(x)} \cdot f'(x) = \left[1 \ln(x+1) + x \cdot \frac{1}{x+1} \cdot 1\right] + \frac{1}{\ln(x-2025)} \cdot \frac{1}{x-2025} \cdot 1$$

Finally, multiply $f(x)$ and $\ln(f(x))$ over to the right side, and substitute their formulas from earlier:

$$\begin{aligned} f'(x) &= f(x) \cdot \ln(f(x)) \cdot \left[\ln(x+1) + \frac{x}{x+1} + \frac{1}{(x-2025)\ln(x-2025)}\right] \\ &= \boxed{(x-2025)^{(x+1)^x} \cdot (x+1)^x \cdot \ln(x-2025) \cdot \left[\ln(x+1) + \frac{x}{x+1} + \frac{1}{(x-2025)\ln(x-2025)}\right]} \end{aligned}$$

[2] These derivative tactics, especially logarithmic differentiation, should really be applied to $\ln|f(x)|$ instead of to $\ln(f(x))$, where the base of the exponential is positive. We ignore this technical issue and just presume all relevant factors are positive.

Points earned: ______
out of a possible 12 points

3. [13 pts] Evaluate the indefinite integral $\int x \arcsin(x)\, dx$.

Solution:

First use integration by parts with $u = \arcsin(x)$ and $dv = x\, dx$. (Inverse trig functions make a better derivative choice than power functions do!) This gives us

$$du = \frac{1}{\sqrt{1-x^2}}\, dx \qquad v = \frac{x^2}{2}$$

and therefore

$$\int x \arcsin(x)\, dx = \frac{x^2}{2} \arcsin(x) - \left(\int \frac{x^2}{2\sqrt{1-x^2}}\, dx \right)$$

Now, to handle the integral in parentheses, we use trigonometric substitution. Choose

$$x = \sin(\theta) \qquad dx = \cos(\theta)\, d\theta \qquad \sqrt{1-x^2} = \sqrt{1-\sin^2(\theta)} = \sqrt{\cos^2(\theta)} = \cos(\theta)$$

Therefore, we obtain

$$\frac{x^2}{2} \arcsin(x) - \left(\int \frac{(\sin\theta)^2}{2(\cos\theta)} \cdot \cos(\theta)\, d\theta \right) = \frac{x^2}{2} \arcsin(x) - \left(\int \frac{1}{2} \sin^2(\theta)\, d\theta \right)$$

Use the half-angle identity $\sin^2(\theta) = (1/2)(1-\cos(2\theta))$, and after integrating, use double-angle $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$:

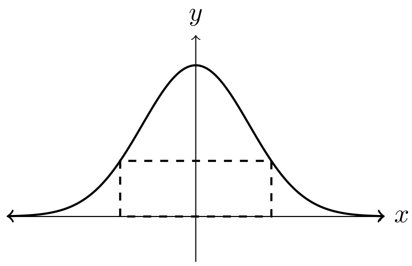
$$\begin{aligned} & \frac{x^2}{2} \arcsin(x) - \left(\frac{1}{2} \int \frac{1}{2}(1-\cos(2\theta))\, d\theta \right) = \frac{x^2}{2} \arcsin(x) - \frac{1}{4} \left(\theta - \frac{\sin(2\theta)}{2} \right) + C \\ = \; & \frac{x^2}{2} \arcsin(x) - \frac{1}{4} \left(\arcsin(x) - \frac{2\sin(\theta)\cos(\theta)}{2} \right) + C = \left(\frac{x^2}{2} - \frac{1}{4} \right) \arcsin(x) + \frac{1}{4} \sin(\theta)\cos(\theta) + C \end{aligned}$$

To get this back in terms of x, we substituted $x = \sin(\theta)$ and also showed $\cos(\theta) = \sqrt{1-x^2}$:

$$\int x \arcsin(x)\, dx = \boxed{\left(\frac{x^2}{2} - \frac{1}{4} \right) \arcsin(x) + \frac{1}{4} x\sqrt{1-x^2} + C}$$

Points earned: ______
out of a possible 13 points

4. [10 pts] A rectangle is created with its base on the x-axis and its top corners intersecting the graph of $f(x) = e^{(-x^2)}$, as pictured. (The exponent on top of e is $-x^2$.)



Show that the absolute maximum area of this rectangle occurs when x is an inflection point of $f(x)$, and determine the value(s) of x.

Solution:

Let (x, y) represent the top-right corner of this rectangle, so $x \geq 0$ and $y = e^{(-x^2)}$.

Since the other top corner is $(-x, y)$, the rectangle width is $2x$ and height is y, so the area of the rectangle is

$$A(x) = 2xe^{(-x^2)} \qquad \text{for all } x > 0$$

To maximize this, we find critical values by taking the derivative and factoring:

$$A'(x) = 2e^{(-x^2)} + 2x\left[e^{(-x^2)} \cdot (-2x)\right] = (2 - 4x^2)e^{(-x^2)} = 0$$

Because exponential functions are never zero, this means $2 - 4x^2 = 0$, so $x^2 = 1/2$. The only positive critical value is $x = \sqrt{1/2} = 1/\sqrt{2}$ aka $x = \sqrt{2}/2$.

Next, let's verify this yields an absolute maximum by studying the signs of $A'(x)$. Note that $e^{(-x^2)}$ is always positive. Since $2 - 4x^2 > 0$ when $x^2 < 1/2$ (meaning $0 < x < \sqrt{1/2}$) and $2 - 4x^2 < 0$ when $x^2 > 1/2$ (so $x > \sqrt{1/2}$), $A'(x)$ changes from positive to negative at $x = \sqrt{1/2}$, so this is a local maximum. However, this is the **only place in the whole domain** $(0, \infty)$ where $A'(x)$ changes sign, so $A(x) < A(\sqrt{1/2})$ for all $0 < x < \sqrt{1/2}$ and $A(\sqrt{1/2}) > A(x)$ for all $x > \sqrt{1/2}$, so we have an **absolute** maximum at $\boxed{x = \sqrt{1/2}}$.[3]

Finally, why is this an inflection point of $f(x)$? Note that $f'(x) = e^{(-x^2)} \cdot -2x = -A(x)$, and therefore $f''(x) = -A'(x)$, so $f''(x) = 0$ at the same places where $A'(x) = 0$. Thus, $x = \sqrt{1/2}$ is also the positive inflection point of $f(x)$ (with negative inflection point $-\sqrt{1/2}$ as well).

[3] If we allowed x to represent either the top-left or top-right corner, we could argue that $x = -\sqrt{1/2}$ is also a location of the absolute maximum.

Points earned: ______
out of a possible 10 points

5. [12 pts] Suppose that the arc length $L(x)$ of an increasing smooth function $f(x)$, starting from $(0, f(0)) = (0, a)$ to $(x, f(x))$, has the form $L(x) = Cx$ where $x \geq 0$ and C is constant. What are the allowable values for the constant C? And what is the general form of $f(x)$?

Solution (credit mostly to Shuzhou Wang):

The arc length of a continuously differentiable function $f(x)$ on the interval $[c, d]$ is given by

$$L(x) = \int_c^d \sqrt{1 + (f'(t))^2}\, dt$$

Therefore, on the interval $[0, x]$ (with $x \geq 0$), our given information tells us

$$L(x) = Cx = \int_0^x \sqrt{1 + (f'(t))^2}\, dt$$

We differentiate both sides with respect to x using the Fundamental Theorem of Calculus (part 1) to get

$$\frac{d}{dx}[Cx] = C = \sqrt{1 + (f'(x))^2}$$

Notice that since C is a positive square root, we must have $C \geq 0$. (Alternately, the arc length must be increasing, so $L'(x) = C \geq 0$.) Solving for $f'(x)$, we get

$$C^2 = 1 + f'(x)^2 \quad \Leftrightarrow \quad f'(x)^2 = C^2 - 1 \quad \Leftrightarrow \quad f'(x) = \sqrt{C^2 - 1}$$

Only the positive square root makes sense here since $f(x)$ is assumed to be increasing, and therefore $f'(x) \geq 0$. In order for square root to be real, we need $C^2 - 1 \geq 0$, so $C^2 \geq 1$. Since we know $C \geq 0$ as well, this means we need $\boxed{C \geq 1}$.

Finally, we have an initial-value problem to solve for $f(x)$, since we have $f'(x)$ and one initial point $(0, a)$. The indefinite integral is

$$f(x) = \int f'(x)\, dx = \int \sqrt{C^2 - 1}\, dx = (\sqrt{C^2 - 1})x + K$$

for some constant K (since $\sqrt{C^2 - 1}$ is just a constant). Plugging in $(0, a)$ yields $K = a$:

$$\boxed{f(x) = (\sqrt{C^2 - 1})x + a}$$

Points earned: ______
out of a possible 12 points

6. [13 pts] Evaluate the definite integral $\displaystyle\int_0^1 \frac{1}{\sqrt{x}+x^{1/3}}\,dx$.

Solution:

The main idea is to use a *rationalizing substitution* u so that the integral becomes a rational function in u. Thus, we want $\sqrt{x} = x^{1/2}$ and $x^{1/3}$ to become positive integer powers of u. To do that, let's use a common denominator for the exponents to write $x^{1/2} = x^{3/6} = (x^{1/6})^3$ and $x^{1/3} = x^{2/6} = (x^{1/6})^2$.

As a result, we substitute

$$u = x^{1/6} \qquad \Leftrightarrow \qquad x = u^6 \qquad dx = 6u^5\,du$$

For the integration limits: when $x = 0$, $u = 0^{1/6} = 0$, and when $x = 1$, $u = 1^{1/6} = 1$. Therefore,

$$\int_0^1 \frac{1}{\sqrt{x}+x^{1/3}}\,dx = \int_{x=0}^{x=1} \frac{dx}{(x^{1/6})^3 + (x^{1/6})^2} = \int_{u=0}^{u=1} \frac{6u^5\,du}{u^3+u^2}$$

By factoring $u^3 + u^2 = u^2(u+1)$, we can cancel u^2 from top and bottom to get $6u^3/(u+1)$ for our new integral. This is an **improper rational function** (with degree on top $\geq$ degree on bottom), so we use long division![4] We skip showing all the steps and obtain

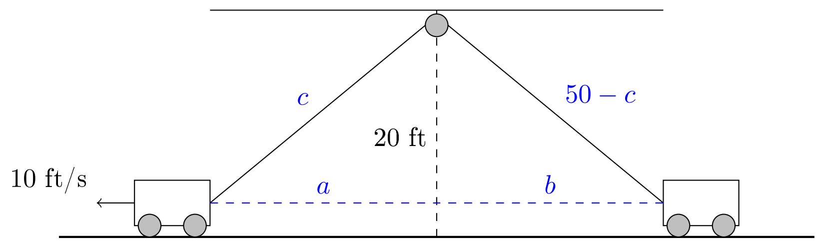
$$\begin{aligned}
\int_0^1 \frac{6u^5}{u^2(u+1)}\,du &= \int_0^1 \frac{6u^3}{u+1}\,du = \int_0^1 (6u^2 - 6u + 6) + \frac{-6}{u+1}\,du \\
&= \left(\frac{6u^3}{3} - \frac{6u^2}{2} + 6u - 6\ln|u+1|\right)\Bigg|_0^1 \\
&= \boxed{(2(1)^3 - 3(1)^2 + 6(1) - 6\ln(2)) - (0 - 0 + 0 - 6\ln(1))} = 5 - 6\ln(2)
\end{aligned}$$

[4] An alternative is to substitute $w = u+1$ aka $u = w-1$, which turns the problem into $6(w-1)^3/w$, and this can be expanded.

Points earned: ______
out of a possible 13 points

7. [15 pts] Two carts are connected by a rope that is 50 ft long.

The rope hangs over a pulley 20 ft off the ground, as shown below. (*Labels added for the solution in blue.*)



The cart on the left moves to the left at a speed of 10 ft/sec. How fast is the cart on the right moving when it is 5 ft to the right of the point on the ground directly underneath the pulley?

Solution (credit mostly to Nur Saglam):

We add labels to this drawing as shown above (drawn in blue to distinguish them from the original drawing). Note that the two hypotenuses must add up to 50 feet of rope, which is why we may label them as c and $50 - c$.

Here, a, b, c are all functions of time. Our given information at this moment is

$$\frac{da}{dt} = +10 \text{ ft/sec} \qquad\qquad b = 5 \text{ ft}$$

(The car on left is moving farther away, so a is growing and hence $a'(t) > 0$.)

Write the Pythagorean Theorem for the right triangles on each side of the pulley:

$$a^2 + (20)^2 = c^2 \qquad\qquad b^2 + (20)^2 = (50 - c)^2$$

Plugging in our known data, we can solve for current values of a and c:

$$50 - c = \sqrt{b^2 + 400} = \sqrt{25 + 400} = \sqrt{425} \quad \Rightarrow \quad c = 50 - \sqrt{425}$$

$$a^2 + 20^2 = c^2 \quad \Rightarrow \quad a^2 = c^2 - 400 = (2500 - 100\sqrt{425} + 425) - 400 = 2525 - 100\sqrt{425}$$

Taking the derivative of the first Pythagorean equation with respect to t:

$$2a\frac{da}{dt} + 0 = 2c\frac{dc}{dt} \qquad \Rightarrow \qquad \frac{dc}{dt} = \frac{a}{c}\frac{da}{dt}$$

By subtracting our two Pythagorean Theorem equations, we end up obtaining an equation which is a bit simpler to differentiate. We can also substitute our dc/dt expression from the line above:

$$(a^2 + 20^2) - (b^2 + 20^2) = c^2 - (50 - c)^2 \quad \Rightarrow \quad a^2 - b^2 = c^2 - (c^2 - 100c + 2500) = 100c - 2500$$

$$\Rightarrow \qquad 2a\frac{da}{dt} - 2b\frac{db}{dt} = 100\frac{dc}{dt} \qquad \Rightarrow \quad 2a\frac{da}{dt} - 2b\frac{db}{dt} = 100\left(\frac{a}{c}\frac{da}{dt}\right)$$

$$\Rightarrow \qquad a\frac{da}{dt} - b\frac{db}{dt} = 50\frac{a}{c}\frac{da}{dt} \qquad \Rightarrow \quad b\frac{db}{dt} = 50\frac{a}{c}\frac{da}{dt} - a\frac{da}{dt} = a\left(\frac{50 - c}{c}\right)\frac{da}{dt}$$

$$\Rightarrow \qquad \frac{db}{dt} = \frac{a}{b}\left(\frac{50 - c}{c}\right)\frac{da}{dt} \qquad \Rightarrow \quad \frac{db}{dt} = \boxed{\frac{\sqrt{2525 - 100\sqrt{425}}}{5}\left(\frac{\sqrt{425}}{50 - \sqrt{425}}\right) \cdot 10} \text{ ft/sec}$$

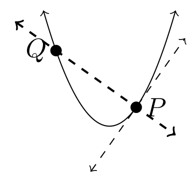
Points earned: ______
out of a possible 15 points

8. [15 pts]

Let $P(a, a^2)$ be a point on the parabola $y = x^2$ with $a \neq 0$.

Suppose that the normal line (perpendicular to the tangent line) to the parabola at P intersects the parabola at a second point Q.

Find the value of a which minimizes the distance from P to Q.



Solution (credit to Nur Saglam for the wonderful algebra organization!):

Say $f(x) = x^2$ with $f'(x) = 2x$.

Given $a \neq 0$, suppose the normal line at $(a, f(a))$ intersects $y = f(x)$ at $(b, f(b))$ where $b \neq a$. The slope of the normal line is

$$m_N = \frac{b^2 - a^2}{b - a} = \frac{(b-a)(b+a)}{b-a} = b + a$$

However, the normal line is also perpendicular to the tangent line at a, so we also have $m_N = -1/f'(a)$. Therefore,

$$a + b = \frac{-1}{2a} \qquad \Rightarrow \qquad b = -a - \frac{1}{2a} \qquad \Rightarrow \qquad a - b = a - \left(-a - \frac{1}{2a}\right) = 2a + \frac{1}{2a}$$

We want to minimize the distance $d = \sqrt{(a-b)^2 + (a^2 - b^2)^2}$, but since d is positive, we can instead minimize the **square** of the distance and get the same critical points! Namely, let's write the square and substitute our expressions for $a - b$ and $a + b$ from above:

$$\begin{aligned} D &= (a-b)^2 + (a^2-b^2)^2 = (a-b)^2 + [(a-b)(a+b)]^2 = (a-b)^2\left[1 + (a+b)^2\right] \\ &= \left(2a + \frac{1}{2a}\right)^2 \left[1 + \left(\frac{-1}{2a}\right)^2\right] = \left(\frac{4a^2+1}{2a}\right)^2 \left[\frac{4a^2+1}{4a^2}\right] \\ D(a) &= \frac{(4a^2+1)^3}{(4a^2)^2} = \frac{(4a^2+1)^3}{16a^4} \end{aligned}$$

Now we compute the derivative $D'(a)$ using the quotient rule, along with the Chain Rule to differentiate $(4a^2+1)^3$ **instead** of expanding! After that, we factor out as much as possible:

$$\begin{aligned} D'(a) &= \frac{[3(4a^2+1)^2 \cdot 8a] \cdot 16a^4 - (4a^2+1)^3 \cdot 64a^3}{(16a^4)^2} \\ &= (4a^2+1)^2 \frac{[24 \cdot 16a^4] - [4a^2+1](64a^3)}{(16a^4)^2} = (4a^2+1)^2 \cdot 64a^3 \cdot \frac{[6a] - [4a^2+1]}{(16a^4)^2} \\ &= (4a^2+1)^2 \cdot \frac{64a^3}{16^2 a^8} \cdot (2a^2 - 1) = \frac{(4a^2+1)^2(2a^2-1)}{4a^5} \end{aligned}$$

Therefore, $D'(a) = 0$ when $(4a^2+1) = 0$ (leading to $a^2 = -1/4$, impossible) or $(2a^2-1) = 0$ (leading to $a^2 = 1/2$). Our critical values are $\boxed{a = \pm\sqrt{1/2}}$. Note that $2a^2 - 1 < 0$ when $0 < a < \sqrt{1/2}$ and $2a^2 - 1 > 0$ when $a > \sqrt{1/2}$, showing that $D'(a)$ changes from negative to positive at $a = \sqrt{1/2}$, and $D(a)$ has a local minimum at $a = \sqrt{1/2}$.

Because $D(a)$ is an even function, symmetry shows that $a = -\sqrt{1/2}$ is also a local minimum (with the same $D(a)$ value!). Therefore, because these are the only local extrema of the graph, with $D(a) \to \infty$ as either $a \to \pm\infty$ or $a \to 0$, we obtain the absolute minimum at $a = \pm\sqrt{1/2}$.