Real Analysis Qualifying Exam

August, 2024

State clearly which theorem you are using. You may quote the conclusion of a part of one problem for another (part of the same or different problem) even if you have not proved the quoted one. Each of the six problems is worth 10 points.

Notation: m is the Lebesgue measure on the set $\mathbb{R}$ of reals, and so is dx.

1. Show that the function $f(x) = \dfrac{1}{1-e^{-x^2}}$ is uniformly continuous outside $(-\delta, \delta)$ for every $\delta > 0$, but it fails to be uniformly continuous on $\mathbb{R}$.

2. Given a sequence of measurable sets $A_1, A_2, \dots$ in $[0,1]$, define

$$\limsup A_n := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n, \quad \liminf A_n := \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n$$

   Show that

$$\liminf A_n \subset \limsup A_n$$

   and that

$$\limsup m(A_n) \leq m(\limsup A_n), \quad m(\liminf A_n) \leq \liminf m(A_n).$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Recall that a point $x \in \mathbb{R}$ is a *critical point* of f if $f'(x) = 0$, and a point $y \in \mathbb{R}$ is a *critical value* of f if $y = f(x)$ for some critical point x. Prove that the set of all critical values of f has Lebesgue measure zero.

   Suggestion: Consider the Mean Value Theorem.

4. Let E be a Lebesgue measurable set with $m(E) < \infty$. For each $x \in \mathbb{R}$, let $E + x = \{y + x : y \in E\}$, and define

$$f(x) = m(E \cap (E + x)).$$

   Show that (a) $f \in L^1(\mathbb{R})$, and (b) $\lim_{|x| \to \infty} f(x) = 0$.

5. For $t \in (0, \infty)$, define $f(t) := \int_{\mathbb{R}} e^{-tx^2} dx$. Show that (a) $f'(t)$ exists, and (b) $f'(t)$ is continuous.

6. For the Lebesgue measure, (a) define $L^\infty(\mathbb{R})$ and $\|f\|_\infty$ for $f \in L^\infty(\mathbb{R})$, and (b) show that $L^\infty(\mathbb{R})$ is a Banach space.