Ph.D. Prelim: Probability Theory, August 2004

1. Show that random variables $\{X_n\}$ and X satisfy $X_n \to X$ in distribution iff
$$E[F(X_n)] \to E[F(X)]$$
for every continuous distribution function F.

2. Let $\{X_n\}$ be iid random variables, $E|X_1| < \infty$, and denote $S_n = X_1 + \cdots + X_n$. Prove that
$$E[X_1 | S_n, S_{n+1}, \ldots] = \frac{S_n}{n} \quad \text{a.s.}$$

3. Prove for iid random variables $\{X_n\}$ with $S_n = X_1 + \cdots + X_n$ that
$$\frac{S_n - C_n}{n} \to 0 \text{ a.s.}$$
for some sequence of constants C_n if and only if $E|X_1| < \infty$.

4. Let $\{X_n\}$ be iid random variables with $E|X_1| < \infty$. Show that
$$\lim_{n\to\infty} \frac{1}{n} E(\max_{1\le k\le n} |X_k|) = 0.$$

5. (a) Quote without proof the Lindeberg-Feller CLT.

   (b) Show that for the sequence $\{X_n\}$ of independent random variables wtih
$$P(X_n = n) = P(X_n = -n) = \frac{1}{2},$$
the CLT holds.

6. If $\{X_n\}$ iid, $EX_1 = 0$, $E(|X_1| \log^+ |X_1|) < \infty$, then $\sum X_n/n$ converges a.s.