Real Analysis Qualifying Examination — Fall 2003

Show work and carefully justify/prove your assertions.

Problem 1. Let X, μ be a finite measure space and f be a non-negative measurable function on X. Prove that f is integrable if and only if the sum:

$$\sum_{n=0}^{\infty} 2^n \mu(E_n) < \infty$$

where $E_n = \{x \in X : \ f(x) \geq 2^n\}$.

Problem 2. Let $f_n(x) \geq f_{n+1}(x) \geq 0$ be a non-increasing sequence of continuous functions defined on the interval [0,1].

a) Prove that if $\lim_{n\to\infty} f_n(x) = 0$ for every $0 \leq x \leq 1$ then the sequence f_n converges uniformly to 0 on the interval [0,1].

b) Does this remain true if the functions f_n are defined on the *open* interval $(0, 1)$?

Problem 3. Let $f(x)$ be a twice continuously differentiable function defined for all real numbers x. Suppose $f(0) = 0$ and f has a local minimum at 0.

Prove that there is a circle through the origin centered on the y- axis , which lies above the graph of f.

Problem 4. Let $f(x)$ be a non-negative Lebesgue integrable function on the interval $(0, 1)$. For $0 < x < 1$ define

$$g(x) = \int_x^1 t^{-1} f(t)\, dt$$

Prove that $g(x)$ is Lebesgue integrable on $(0, 1)$ and

$$\int_0^1 g(x)\, dx = \int_0^1 f(x)\, dx$$

Problem 5. Let f be a bounded and measurable function on $[0, \infty)$. For $0 \leq x < \infty$ define

$$F(x) = \int_0^x f(t)\, dt$$

for $0 \leq x < \infty$.

a) Show that there exists a constant $M > 0$ such that:
$|F(x) - F(y)| \leq M|x - y|$

b) Prove that there exist a constant C such that $m(F(E)) \leq Cm(E)$ for every Lebesgue measurable subset E of $[0, \infty)$. Here $m(E)$ denotes the Lebesgue measure of the set E and $F(E) = \{F(x) : \ x \in E\}$.