Ph.D. Comprehensive Examination on Algebra

Fall 2003

You have three hours to complete this exam. Please write your solutions in a clear and concise fashion.

1. Suppose \( R \) is a commutative ring with identity where \( 1_R \neq 0 \). Prove that the following are equivalent.
   (i) \( R \) is a field;
   (ii) \( R \) has no proper ideals;
   (iii) \( 0 \) is a maximal ideal in \( R \);
   (iv) every nonzero homomorphism of rings \( R \to S \) is a monomorphism.

2. State the three Sylow theorems. Prove that there are no simple groups of order 36.

3. Compute the Galois group of \( x^4 + 1 \) over \( \mathbb{Q} \).

4. Let \( A \) be a symmetric real \( n \times n \) matrix. Show that all the eigenvalues of \( A \) must be real and that the eigenvectors corresponding to the different eigenvectors are orthogonal.

5. Determine the Jordan canonical form of the following matrix:

\[
\begin{pmatrix}
-1 & 1 & 0 & 0 & 0 \\
-4 & 3 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.
\]

6. Prove that every group of order \( p^2q \) where \( p \) and \( q \) are primes is solvable.

7. Let \( K \) be a field. Prove that the polynomial ring \( K[x] \) is a principal ideal domain.

8. Let \( R \) be a ring and \( f : M \to N \) and \( g : N \to M \) be \( R \)-module homomorphisms such that \( g \circ f = \text{id}_M \). Show that \( N \cong \text{Im} f \oplus \text{Ker} g \).