Qualifying Exam - Analysis

Fall 2002

Problem 1. Let $f(x)$ be a continuous function which maps the interval [0,1] into itself.

a) Prove that there exists an x between 0 and 1 for which $f(x) = x$.

b) Prove that if in addition $f(0) = f(1) = 0$ and $f(1/2) = 1$ then there exist $0 \leq x < y \leq 1$ for which $f(x) = y$ and $f(y) = x$.

Problem 2. Let E be a Lebesgue measurable subset of the real line of positive measure.

a) Give the definition of x being a Lebesgue point of E and show that for every $\epsilon > 0$ and $N > 0$ there exists an $n \geq N$ and an interval I of length 2^{-n} such that:

$$m(E \cap I) \geq (1 - \epsilon)m(I)$$

where $m(E)$ denotes the Lebesgue measure of the set E.

b) Prove that if $E + 2^{-n} = E$ for every rational r then either $m(E) = 0$ or $m(\Re/E) = 0$. Here $E + r = \{x + r : \ x \in E\}$ is the translate of the set E by r.

Problem 3. Let $f(x) = \sum_{n=0}^{\infty} 3^{-n} \sin(9^n \pi x)$.
Show that $f(x)$ is uniformly continuous. Find a point where it is not differentiable.

Problem 4. State Fatou's lemma. Give an example of a sequence of functions satisfying the hypothesis of Fatou's lemma, for which strict inequality holds in the conclusion.

Problem 5. Prove that any two norm on a finite dimensional linear space X is equivalent, that is there are constants $0 < c < C$ such that for all $x \in X$:

$$c||x||_1 \leq ||x||_2 \leq C||x||_1$$

Prove that any finite dimensional subspace of a complete normed linear space X is closed and nowhere dense (completeness means that of the associated metric $d(x, y) = ||x - y||$).

If the space X is not finite dimensional can it be a countable union of finite dimensional subspaces ? Can it have a countable basis as a linear space ?

Problem 6. Consider the convolution:

$$f * g(x) = \int_{\Re} f(x-t)g(t)\,dt$$

where f is continuous of compact support and $g = \chi_{[a,b]}$ is the indicator function of the interval [a,b].

Show that $(f * g)'(x) = f(x-a) - f(x-b)$ and thus $f * g$ is continuously differentiable of compact support.

Hint: Change variables $t \to t - h$ in evaluating $f * g\,(x + h)$ and use the mean value theorem to calculate the limit $1/h(f * g\,(x+h) - f * g\,(x))$ as $h \to 0$.

Problem 7. Compute the integral:

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx$$

Problem 8. State Schwarz' lemma.

Prove that if f and g are holomorphic mappings of the unit disc U into an open domain Ω, f is one-to-one, $f(U) = \Omega$ and $f(0) = g(0)$, then

$$g(D(0,r)) \subseteq f(D(0,r))$$

for all $0 < r < 1$, wher $D(0,r)$ is the open disc of radius r.

Problem 9. State Rouche's theorem. Prove that

$$\max_{|z|=1} |a_0 + a_1 z + \ldots a_{n-1} z^{n-1} + z^n| \geq 1$$

Problem 10. Let Ω be the upper half of the unit disc U. Find the conformal mapping f of Ω onto U that carries $\{-1, 0, 1\}$ to $\{-1, -i, 1\}$.

Hint: Try $f = t_\alpha \circ s \circ t_\beta$, where $s(z) = z^2$ and $t_\alpha = \frac{z-\alpha}{1-\alpha z}$ is a linear fractional transformation.