Algebra Qualifying Exam, Fall 2002

The questions are all of equal value.

1. Let R be a commutative ring. Prove that if every ascending chain of ideals in R stabilizes, then every ideal in R is finitely generated.

2. i) Let p be a prime number and let S be a group whose order is a power of p. Let X be a finite set on which S acts. Prove that the number of elements of X is congruent modulo p to the number of fixed points.

   ii) Let S and S' be Sylow p-subgroups of a finite group G. By letting S' act on G/S, deduce from part i) that S and S' are conjugate.

3. Let M denote the $\mathbb{Z}$-module with two generators α_1 and α_2, subject to the relations

$$\begin{aligned} 111\alpha_1 + 63\alpha_2 &= 0 \\ 6\alpha_1 + 3\alpha_2 &= 0. \end{aligned}$$

   Express M as a direct sum of cyclic modules.

4. Let A be a square matrix over an algebraically closed field K, such that A has only one eigenvalue. Prove: The minimal polynomial of A equals the characteristic polynomial of A if and only if every matrix that commutes with A is a polynomial in A.

5. Let a, b, c be rational numbers, and let E be the splitting field over $\mathbb{Q}$ of the polynomial $f(x) = x^3 - ax^2 + bx - c$. Let R be the algebra over $\mathbb{Q}$ generated by three letters $\alpha_1, \alpha_2, \alpha_3$, subject to the relations

$$\alpha_1 + \alpha_2 + \alpha_3 = a, \quad \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 = b, \quad \alpha_1\alpha_2\alpha_3 = c.$$

   Prove that $\mathrm{Gal}_{\mathbb{Q}}(E) = \Sigma_3$ if and only if R is a field. (Σ_3 denotes the symmetric group on three letters.)
   (Hint: Write down a homomorphism from R to E.)

6. Consider the rings $R = \mathbb{Q}[x]/(x^5)$ and $S = \mathbb{Q}[x]/(x^3)$. Regard S as an R-module via the homomorphism $R \to S$ sending x to x. Prove that if M is a finitely generated R-module, and $M \otimes_R S = 0$, then $M = 0$.
   (Hint: Show that the natural map $M \to M/xM$ factors through $M \otimes_R S$.)