Analysis Prelim Exam - August 2001

1. Suppose $K : [0,1] \times [0,1] \to \mathbb{R}$ is continuous. For any function f on $[0, 1]$ set

$$\mathcal{K}f(x) = \int_0^1 K(x,y)f(y)dy.$$

   a) If $f \in \mathcal{L}^p[0,1]$ for some $p \geq 1$, prove $\mathcal{K}f \in \mathcal{C}[0,1]$.
   b) If $\{f_n : n \geq 1\}$ is a Cauchy sequence in $\mathcal{L}^1[0,1]$, prove $\{\mathcal{K}f_n : n \geq 1\}$ is a Cauchy sequence in the supnorm of $\mathcal{C}[0,1]$.

2. Suppose $\{f_n : n \geq 1\}$ and $\{\frac{df_n}{dx} : n \geq 1\}$ are sequences in $\mathcal{C}[0,1]$. If $\{f_n : n \geq 1\}$ converges uniformly to f and $\{\frac{df_n}{dx} : n \geq 1\}$ converges uniformly to g, prove f is differentiable and $\frac{df}{dx} = g$.

3. Suppose f is a measurable function on $\mathbb{R}^2$, both $\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} f(x,y)dx)dy$ and $\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} f(x,y)dy)dx$ exist and are finite, and $\{(x,y) : f(x,y) \neq 0\}$ has nonzero measure.
   a) If no additional conditions are given, construct an f such that

$$\int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} f(x,y)dx)dy \neq \int_{-\infty}^{\infty}(\int_{-\infty}^{\infty} f(x,y)dy)dx.$$

   b) Give an additional condition on f that guarantees the two integrals are equal.

4. Let μ be an outer measure on $\mathbb{R}^n$. Prove open sets are measurable with respect to μ if and only if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $\text{dist}(A,B) > 0$.

5. Suppose p is a monotonically decreasing positive valued function on $[0,\infty)$, with $p(0) < \infty$. If $\lim_{x\to\infty} p(x) = 0$, prove

$$\lim_{R\to\infty} \int_0^R p(x)\cos \lambda x dx$$

   exists for any $\lambda > 0$.

6. Evaluate $\int_{-\infty}^{\infty} \frac{\sin^3 x}{x^3}dx$. Verify all your steps.

In the following exercises $\Omega = \{z : |z| < 1\}$ and $\mathbf{H} = \{z : \text{Im} z > 0\}$.

7. Construct a one to one conformal mapping of $\Omega \cap H$ onto Ω.

8. Let $\{f_n : n \geq 1\}$ be a sequence of holomorphic functions on Ω. Suppose $\{f_n : n \geq 1\}$ converges uniformly on compact subsets of Ω to a function f.
   a) Prove f is holomorphic on Ω
   b) If D is a closed disk in Ω and f is nowhere 0 on ∂D, then there is an $N > 0$ such that for all $n > N$ # zeros of f in D = # of zeros of f_n in D.

9. Determine the Laurent expansion of $f(z) = \dfrac{1}{z(z-1)(z-2)}$ in each of the following regions:
   a) $\{z : 0 < |z| < 1\}$;
   b) $\{z : 1 < |z| < 2\}$; and
   c) $\{z : 2 < |z|\}$.
10. Let $u : \Omega \to \mathbb{R}$ be a C^2-harmonic function. That is,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Construct a real valued function v on Ω such that $f = u + iv$ is holomorphic on Ω.