Analysis Preliminary Examination — Fall 2000

Show work and carefully justify/prove your assertions.

1. (1) Let p^* be a limit point of a subset A of $\mathbb{R}^N$. Show that each ball $B(p^*, r)$ (where $r > 0$) around p^* contains infinitely many points of A;

   (2) Let A and B be two disjoint compact subsets of a metric space with metric d. Show that there exist $a \in A,\ b \in B$ such that

   $$d(a,b) = \inf\{d(x,y) \mid x \in A, y \in B\}.$$

2. Let $R(x)$ be the function on $[0,1]$ defined by

   $$R(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ is rational (integers } m \text{ and } n \text{ have no common factors)} \end{cases}$$

   Show that $R(x)$ is Riemann integrable on $[0,1]$ and find the value of the integral.

3. Let $f(x)$ be differentiable on $[0,1]$. Show that its derivative $f'(x)$ is measurable on $[0,1]$.

4. Let $f(x) \in L^1(\mathbb{R})$ and let $g(x)$ be a function on $\mathbb{R}$ with continuous first order derivative. Suppose that $g(x)$ vanishes outside a bounded closed interval. Define a new function $h(x)$ by

   $$h(x) = \int_{\mathbb{R}} f(x-t)g(t)dt.$$

   Show that $h(x)$ is differentiable on $\mathbb{R}$.

5. Let $C[0,1]$ be the space of all complex valued continuous functions on the unit interval endowed with the norm

   $$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|.$$

   Let $C^1[0,1]$ be the space of all complex valued functions with continuous first order derivative, endowed with the norm

   $$\|f\| = \|f\|_\infty + \|f'\|_\infty.$$

   Let B be the unit ball in $C^1[0,1]$. Show that the closure of B in $C[0,1]$ is compact.

   Hint: What are some equivalent conditions for compact sets in a metric space?

6. (1) Let z_k $(k = 1, \cdots, n)$ be complex numbers lying on the same side of a straight line passing through the origin. Show that

   $$z_1 + z_2 + \cdots + z_n \neq 0, \quad 1/z_1 + 1/z_2 + \cdots + 1/z_n \neq 0.$$

   (2) Let $f(z) = z + 1/z$. Describe the images of both the circle $|z| = r$ of radius r $(r \neq 0)$ and the ray $\arg z = \theta_0$ under f in terms of well known curves.

7. Let f be analytic on a region R. Suppose $f'(z_0) \neq 0$ for some $z_0 \in R$. Show that if C is a circle of sufficiently small radius centered at z_0, then

$$\frac{2\pi i}{f'(z_0)} = \int_C \frac{dz}{f(z) - f(z_0)}.$$

8. Let A be the intersection of the disk $|z+i| < \sqrt{2}$ with the open upper half plane. Find a bijective conformal map from A to the unit disk.

9. Consider a one-to-one analytic mapping $w = u + iv = f(z) = \sum_{n=0}^{\infty} c_n z^n$ from the open unit disk $|z| < 1$ in the xy-plane ($z = x + iy$) to a region in the uv-plane. For $0 < r < 1$, let D_r be the disk $|z| < r$.

   (1) Show that the area $\int\int_{f(D_r)} dudv$ of $f(D_r)$ is finite and is given by $\pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}$;

   (2) Give an example of f that is analytic in $|z| < 1$ but the area of $f(D_1)$ is infinite.

10. Let $f(z)$ be an analytic function on $\mathbb{C}\backslash\{z_0\}$, where z_0 is a given number. Assume that $f(z)$ is bijective from $\mathbb{C}\backslash\{z_0\}$ onto its range, and that $\lim_{z\to\infty} f(z)$ exists and is finite. Show that $f(z)$ is a fractional linear transformation. Hint: Consider the Laurent series expansion of $f(z)$.