P.H.D. Preliminary Examination Nunerical Analysis
September 11, 1997

1) Discuss the calculation of of e^{-x} for $x > 0$ from the series

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

Suggest a better way assuming that the system function for e^x is not available.

2) Suppose that r is a root of multiplicity $m > 1$ of $f(x)$. Show that the following modified Newton's method

$$x_{n+1} = x_n - \frac{mf(x_n)}{f'(x_n)}$$

converges quadratically to r for initial guess x_0 sufficiently close to r.

3) Define the condition number, $K(A)$, of a $N \times N$ matrix A relative to a subordinate matrix norm $\|\cdot\|$. Let

$$A = \begin{bmatrix} 10^{10} & 0 \\ 0 & 10^{-10} \end{bmatrix}$$

and compute $K(A)$ relative to the $\|\cdot\|_\infty$ norm. Can one accurately solve the equation $Ax = b$?

4) If $S(x)$ is a first-degree spline that interpolates $f(x)$ at a sequence of knots $0 = t_0 < t_1 < \cdots < t_n = 1$, what is $\int_0^1 S(x)dx$?

5) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 12 \\ 3 & 12 & 27 \end{bmatrix}$$

Compute the Cholesky decomopsition of A. That is, compute the factorization

$$A = LL^T$$

with L a lower triangular matrix with positive diagonal.

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6) Show that if an $N \times N$ matrix A is strictly diagonally dominant then the Jacobi iteration converges to the solution of $Ax = b$ for any initial guess x_0.

7) Let $p_n(x)$ be the polynomial of degree n interpolating a C^∞ function $f(x)$ at the $n+1$ equally spaced points

$$x_i = a + i(\frac{b-a}{n}), i = 0, 1, \dots, n$$

Does $p_n(x)$ always converge to $f(x)$ on $[a, b]$? Explain your answer.

8) Let T be the triangle with vertices $V_1 = (0,0), V_2 = (1,0)$, and $V_3 = (0,1)$. Find the barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$ of any point (x, y) inside T. Let $p_3(x, y)$ be the polynomial of total degree ≤ 3 whose 10 B-net coefficients are all equal to 1. What is the value of $p(\frac{1}{4}, \frac{1}{4})$?

9) Consider the boundary value problem

$$-u''(t) = f(t),\ 0 < t < 1$$
$$u(0) = 0,\ u(1) = 0$$

An approximate solution of the above problem is given by

$$u_n(t) = \sum_{i=1}^{n-1} \alpha_i \phi_i(t)$$

with $\phi_i(t)$ the piecewise linear (with respect to the uniform partition $t_j = \frac{j}{n}$ of $[0, 1]$) function defined by the equation

$$\phi_i(t_j) = \delta_{i,j}.$$

Set up the system of linear equations used to determine the unknown coefficients α_i.

10) Use the data given in the following table:

x	f(x)
0	1.0
0.25	.93941306
0.5	.77880078
0.75	.56978282
1.0	.36787944

to compute your best possible estimate of $\int_0^1 f(x)dx$.