Algebra Preliminary Examination– 1995

1. Let $GL_2(q)$ denote the group of invertible 2×2 matrices with entries in the field $\mathbb{F}_q$. Prove that there are $q+1$ one-dimensional subspaces of $\mathbb{F}_q^2$ and that the map $GL_2(q) \to S_{q+1}$ defined by $T \to G_T$ where $G_T(\langle v \rangle) = \langle Tv \rangle$, for $0 \neq v \in \mathbb{F}_q^2$ is a homomorphism from $GL_2(q)$ to S_{q+1}. Identify the kernel of this map. When $q = 2, 3,$ and 4 identify the image as a subgroup of S_{q+1}.

2. Let η be a primitive 9th root of 1. Let G be the group of automorphisms of $\mathbb{Q}(\eta)$. Show that G permutes the powers of η, and identify G. Find α, β in $\mathbb{Q}(\eta)$ such that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2, [\mathbb{Q}(\beta) : \mathbb{Q}] = 3$. Identify the groups of automorphisms of $\mathbb{Q}(\alpha)$ and $\mathbb{Q}(\beta)$, and find the minimal polynomial of α and β.

3. (a) Let P be a p-Sylow subgroup of a finite group G, and let $N = N_G(P)$ be its normalizer. Prove that $N_G(N) = N$.

   (b) Classify all finite groups of order 55.

4. Prove that a Euclidean domain is a Unique Factorization Domain.

5. Prove that, as a $\mathbb{Z}$-module, $\mathbb{Q}$ is flat but not projective. (Recall that a right R-module M is flat if $M \otimes_R -$ is an exact functor.)

6. Let M be a $\mathbb{Q}[x]$-module with generators v_1, v_2, v_3, v_4, v_5 and relations: $xv_1 = 3v_1 + v_2$; $xv_2 = -v_1 + v_2$; $xv_3 = v_3 + v_5$; $xv_4 = 2v_4$; $xv_5 = v_5$. Show how M can be written as a direct sum of $\mathbb{Q}[x]$-modules each of which is isomorphic to $\mathbb{Q}[x]/(f(x))$ for some $f(x) \in \mathbb{Q}[x]$ by explicitly finding the polynomials $f(x)$. Find the invariant factors and elementary divisors of M and the minimal polynomial of the corresponding linear transformation.

7. (a) Let α and β be complex numbers which are algebraic over $\mathbb{Q}$. Prove that $\alpha + \beta$ is also algebraic over $\mathbb{Q}$.

   (b) Prove that the multiplicative group of non-zero elements of a finite field is cyclic.