Algebra Preliminary Examination

September 1993

1. Let $G$ be a group that contains a subgroup $H$ of index $n$. Show that $G$ contains a normal subgroup $K$ lying in $H$ such that $G/K$ is a finite group of order dividing $n!$.

2. Let $S_4$ denote the permutation group on 4 letters.
   (a) How many non-isomorphic 2-Sylow subgroups does $S_4$ have? Justify your answer.
   (b) How many 2-Sylow subgroups are there in each isomorphism class of 2-Sylow subgroups of $S_4$. Justify your answer.

3. Describe the Galois group of the splitting field over $\mathbb{Q}$ of the polynomial $x^6 + 1$.

4. Give the definition of a separable field extension. Give an example of a finite field extension $F/K$ that is not separable. Justify your answer.

5. In the ring $\mathbb{Z}[x]$, consider the ideal $I$ generated by the elements 2 and $x^2 + x + 1$.
   (a) How many elements does the quotient ring $\mathbb{Z}[x]/I$ contain? Justify your answer.
   (b) Is the ideal $I$ a maximal ideal? Justify your answer.

6. (a) Give the definition of a projective module. Give an example of a commutative ring $A$ and an $A$-module $M$ ($M \neq 0$) that is not projective. Justify your answer.
   (b) Give the definition of a injective module. Give an example of a commutative ring $A$ and an $A$-module $M$ ($M \neq 0$) that is not injective. Justify your answer.

7. Let $A$ be a local domain with maximal ideal $\mathcal{M}$. ($A$ is also assumed to be commutative with the identity element.)
   (a) State Nakayama’s lemma for $A$.
   (b) Suppose now that $\mathcal{M}$ is a principal ideal. Show that if $A$ is noetherian, then
      $$\bigcap_{i=1}^{\infty} \mathcal{M}^i = (0).$$

8. (a) Let $V$ be a non-zero vector space over $\mathbb{C}$, with a positive definite hermitian form $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{C}$. Let $A : V \to V$ be a hermitian map. Show that $V$ has an orthogonal basis consisting of the eigenvectors of $A$.
   (b) Let $A \in M_n(\mathbb{C})$ be a hermitian matrix. Does there exist a matrix $B \in M_n(\mathbb{C})$ such that $B^n = A$? Justify your answer.