Ph.D. Preliminary Exam in Complex Analysis, Spring 1998.

1. Let

$$V = \{f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \mid f(x, y) = (a_1x^2 + b_1xy + c_1y^2, a_2x^2 + b_2xy + c_2y^2), \\ a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}\}\ .$$

a) Identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, $(x, y) \mapsto x + iy$, give a basis over $\mathbb{R}$ for the subspace of V consisting of holomorphic functions.

b) Give a basis over $\mathbb{C}$ for the subspace of V consisting of holomorphic functions.

2. Use the residue theorem to compute $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2} dx$.

3. Given $a > 0$, let C_a denote the circle with diameter along the y-axis, passing through 0 and ia. Let R denote the region inside C_2 and outside C_1. Give a 1-1 conformal mapping from R onto the upper half-plane.

4. a) Let γ denote the circle centered at 0 with radius 7, oriented counterclockwise. Compute $\displaystyle\int_\gamma \frac{dz}{\sqrt{1-z^2}}$, with $\sqrt{1-z^2}$ normalized to be positive at $7i$.

b) Let $\mathcal{U} \subset \mathbb{C}$ denote the complement of the set $\{\ x \in \mathbb{R} \mid |x| \geq 1\ \}$. Show that there is a branch of $\arcsin(z)$ defined on $\mathcal{U}$, i.e., a function $g(z)$ such that $\sin(g(z)) = z$.

5. Given an integer $n > 1$, let $p(z) = z^n - z - 1$, and let r denote the unique positive root of $p(z)$. Show that all the roots of $p(z)$ lie in the disk $|z| \leq r$.

6. a) Establish the identity $\pi^2 \csc^2(\pi z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

b) Compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

7. a) Say what it means for a family of analytic functions on a region G to be *normal.*

b) Let D denote the open unit disk, and let $\mathcal{F}$ be a family of analytic functions on D. Assume there is an analytic function $g(z)$ defined on D, such that for all integers $m \geq 0$ and all $f \in \mathcal{F}$, $|f^{(m)}(0)| \leq |g^{(m)}(0)|$. Prove that $\mathcal{F}$ is normal.

8. Let $\Lambda \subset \mathbb{C}$ be a lattice. Let $w \in \mathbb{C}$ and let φ be a harmonic function defined on the complement of $w + \Lambda$ in $\mathbb{C}$. Assume φ is periodic with respect to Λ.

a) Prove that if φ has at worst a logarithmic singularity at w, i.e.,

$$\varphi(z) = c \log((z-w)(\bar{z}-\bar{w})) + \psi ,$$

where c is a constant and ψ is continuous at w, then φ is constant. (Consider $\frac{\partial \varphi}{\partial z}$.)

b) Prove that there exist holomorphic functions f and g defined on the complement of $w + \Lambda$ in $\mathbb{C}$, such that $\varphi = f + \bar{g}$.

c) Prove or disprove: If φ has at worst a first order pole at w, i.e.,

$$\varphi(z) = \frac{a}{z-w} + \frac{b}{\bar{z}-\bar{w}} + c \log((z-w)(\bar{z}-\bar{w})) + \psi ,$$

where a, b and c are constants and ψ is continuous at w, then φ is constant.