PhD. prelim in Complex Analysis, Spring 1997.

1. Prove any form of the Cauchy-Goursat theorem; (eg. a function holomorphic in an open set in \( \mathbb{C} \) which contains the boundary \( \gamma = \partial \Delta \) and interior of a triangle \( \Delta \), has zero integral around \( \gamma \)).

2. Prove the following form of the "Casorati-Weierstrass" theorem: if \( f: \mathbb{C} \to \mathbb{C} \) is a holomorphic entire function, and if there is a neighborhood \( U = \{ z : |z| > r > 0 \} \) of infinity whose image \( f(U) \) is not dense in \( \mathbb{C} \), then \( f \) is a polynomial.

3. Prove that a sequence of functions holomorphic in an open set \( U \), and which converges uniformly on all closed discs in \( U \), has a limit which is holomorphic in \( U \).

4. Use residues to calculate the real integral \( \int_{-\infty}^{\infty} \frac{dx}{1+x^4} \), and justify your calculation.

5. a) Let \( A = \{ z : \text{Im}(z) > 0 \text{ and } -\pi/2 < \text{Re}(z) < \pi/2 \} \). Find an explicit formula for a (one to one) conformal mapping of the region \( A \) onto the interior of the unit circle.
   b) Let \( B = \{ z : \text{Re}(z) > (\text{Im}(z))^2 - 1 \} \) be the region on the "right side" of the parabola \( x = y^2 - 1 \). Prove or disprove: for every pair of distinct points \( \alpha, \beta \) in \( B \), there is a (one to one) conformal mapping of the region \( B \) to itself, taking 0 to \( \alpha \), and 1 to \( \beta \).

6. (a) Let \( \Omega = \{ n + mi \text{, for all integers } n,m \} \) denote the lattice of "Gaussian integers". Let \( f \) be a complex valued function holomorphic at all \( z \) not belonging to \( \Omega \), and assume that \( f(z) = f(z+\omega) \) for all \( z \) in \( \mathbb{C} - \Omega \), and all \( \omega \) in \( \Omega \). If at \( z = 0 \), \( f \) is either holomorphic or has at worst a simple pole, prove \( f \) is constant.
   (b) Construct a non constant meromorphic function \( g \) on \( \mathbb{C} \), such that \( g(z) = g(z+\omega) \) for all \( z \) in \( \mathbb{C} - \Omega \), and all \( \omega \) in \( \Omega \).

7. Classify all holomorphic automorphisms \( f: \mathbb{C} \cup \{ \infty \} \to \mathbb{C} \cup \{ \infty \} \) of the Riemann sphere.

8. Construct an entire function with simple zeroes at the (positive) square roots of the positive integers, \( \{ n^{1/2} \} \), \( n > 0 \), and no other