Complex Analysis Premlim

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Instructions. Use your own paper to work the problems.

1. Suppose $f : \mathbf{C} \to \mathbf{C}$ is differentiable at z in the sense that there exists an $\mathbf{R}$-linear map $T : \mathbf{C} \to \mathbf{C}$ such that

$$\lim_{w \to 0} \frac{|f(z+w) - f(z) - T(w)|}{|w|} = 0$$

Show that the following four statements are equivalent.

(a) f satisfies the Cauchy-Riemann equations at z.

(b) $iT(w) = T(iw)$ for all $w \in \mathbf{C}$.

(c) T is complex linear.

(d) f is holomorphic at z in the sense that the complex derivative

$$f'(z) = \lim_{w \to 0} \frac{f(z+w) - f(z)}{w}$$

exists.

(e) When the above hold show that the Jacobian of f at z, f being viewed as a map of $\mathbf{R}^2$ to $\mathbf{R}^2$, is $|f'(z)|^2$.

2. Find all points at which the function f given by

$$f(x+iy) = x^3 + y + ixy$$

is holomorphic.

3. Suppose F is holomorphic with $f = F'$ continuous at each point of an open set D.. If γ is a piecewise smooth curve in D, show that the integral of f over γ depends only on the endpoints of γ.

4. Suppose D is an open set with the following property: there is a point $z_0 \in D$ such that D contains the line segment going from z_0 to z for every $z \in D$. If f is continuous on D and the integral of f over every triangle in D is 0, prove that f has an anti derivative on D.

5. Consider the power series $\sum_{n=0}^{\infty} a_n z^n$.

   (a) If it converges for some $z_0 \neq 0$, give a direct proof that it converges absolutely and uniformly in the set $|z| \leq r$ for every $r < |z_0|$.

   (b) Show there is a unique R (the radius of convergence for the power series) with $0 \leq R \leq \infty$ such that

$$\sum_{n=0}^{\infty} |a_n|\,|z|^n < \infty, \quad |z| < R$$

$$\sum_{n=0}^{\infty} |a_n|\,|z|^n = \infty, \quad |z| > R$$

6. Suppose f is continuous on the circle $\gamma(\theta) = e^{i\theta}, \quad 0 \leq \theta \leq 2\pi$. Set

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z}dw, \quad |z| < 1$$

and for an integer n set

$$f^\wedge(n) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta}d\theta$$

   (a) Give a direct proof that

$$F'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w - z)^2}dw, \quad |z| < 1$$

(b) Use the identity

$$\frac{1}{w-z} = \sum_{k=0}^{n} \frac{z^k}{w^{k+1}} - \frac{(z/w)^{n+1}}{w-z}, \quad z \neq w$$

to get an explicit estimate for

$$\left| F(z) - \sum_{k=0}^{n} f^{\wedge}(k) z^k \right|$$

showing that the series converges uniformly to F on each closed disk $|z| \leq r < 1$.

(c) Compute F when $f(w) = \overline{w}.$

(d) If $f(w) = \sum_{n=0}^{\infty} a_n w^n$, $|w| < R$, R being the radius of convergence for the power series, and $1 < R$, find a formula for the coefficients a_n, and show that $F = f$ on the disk $|z| < 1$.

7. Let f be a non-constant holomorphic function defined on an open connected set U.

(a) Show that the zeros of f are isolated.

(b) Suppose D is a closed disk contained in U and that w is a complex number such that $f(z) \neq w$ for all z in ∂D. Let

$$W = \int_{\partial D} \frac{f'(z)}{f(z) - w} dz$$

Show that $w \in f(D)$ iff $W \neq 0$; if $w \in f(D)$ and $z_1, \ldots, z_k$ are the distinct zeros of $f(z) - w$ in D and z_j is a zero of order m_j show that

$$W = \sum_{j=1}^{k} m_j$$