COMPLEX ANALYSIS PRELIMINARY EXAMINATION
May 8, 1995

N.B. Throughout this exam, let

$$\Delta = \{z : |z| < 1\},$$
$$T = \{z : |z| = 1\}, \text{ and}$$
$$\overline{\Delta} = \Delta \cup T = \{z : |z| \leq 1\}.$$

Work all **eight** problems.

1. (10 points) Give the Laurent series of $f(z) = \dfrac{e^z}{z - z^2}$ converging on

   a. $\{z : 0 < |z| < 1\}$

   b. $\{z : 1 < |z|\}$

2. (15 points) Evaluate by contour integration:

$$\int_0^\infty \frac{x \sin x}{x^2 + 4}\, dx .$$

Be sure to provide sufficient detail and justification.

3. (10 points) Give a conformal map from the region $D = \{z : 0 < \mathcal{Re}\, z < \frac{\pi}{2},\ \mathcal{Im}\, z > 0\}$ to the upper half plane.

4. (15 points) Suppose f is analytic on a region $\Omega \subset \mathbb{C}$ containing the unit disk Δ. Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \Delta$ and assume that this series has radius of convergence 1.

   a. Give a self-contained proof that there is at least one point $z \in T$ so that f is **not** analytic at z.

   b. Give examples of the following:

      (i) such a series convergent everywhere on T

      (ii) such an f analytic at all but **two** points of T

      (iii) such an f analytic at $z_o \in T$ with $\sum_{n=0}^{\infty} a_n z_o^n$ divergent

5. (10 points) Suppose f is analytic on a simply connected domain Ω. If C is a simple closed curve lying in Ω and f is one-to-one on C, prove that f is one-to-one on the interior of C as well.

6. (15 points) Suppose $\mathcal{F}$ is a collection of analytic functions $f : \Omega \to \mathbb{C}$. Suppose that for each compact set $E \subset \Omega$ there is a constant M_E so that

$$\sup_{z \in E} |f(z)| \le M_E \quad \text{for all } f \in \mathcal{F} .$$

Give a self-contained proof that for each compact set $E \subset \Omega$ there is a constant N_E so that

$$\sup_{z \in E} |f'(z)| \le N_E \quad \text{for all } f \in \mathcal{F} .$$

7. (10 points) Do ONE of the following problems:

a. Let $\Omega \subset \mathbb{C}$ be a domain. Suppose a continuous function $u \colon \Omega \to \mathbb{R}$ has the mean value property, i.e., for each $a \in \Omega$ and all $r > 0$ sufficiently small,

$$\frac{1}{2\pi} \int_0^{2\pi} u(a + re^{it})\, dt = u(a) .$$

Prove u is harmonic in Ω.

b. Prove that the function $f(z) = \sqrt{z^2 - 1}$ has an analytic branch on $\mathbb{C} - [-1, 1]$. Let γ denote the circle $|z| = 5$, oriented counterclockwise. Evaluate

$$\int_\gamma f(z)\, dz .$$

8. (15 points) Do ONE of the following problems:

a. Classify the one-to-one entire functions.

b. Suppose $f \colon \overline{\Delta} \to \mathbb{C}$ is continuous, analytic on Δ, and $|f(z)| = 1$ on $|z| = 1$. Prove that f extends to a rational function on $\mathbb{C}$.