Comp V

COMPLEX VARIABLES
PRELIMINARY EXAMINATION
SPRING 1993

Work any 8 of the following problems.

1) Evaluate each of the following expressions. Express your answer in the form $a+ib$. Whenever the expression is multiple valued, be sure to give all of the values.

1. $\log(2+i)$
2. $(1+i)^{(1+i)}$
3. $\sqrt[4]{1-i}$
4. $\arcsin(1+i)$

2) Prove that the series

$$\sum_{n=1}^{\infty} \frac{\sin(nz)}{2^n}$$

converges uniformly on compact subsets of the region $\{z : Im(z) < \ln(2)\}$.

3) Prove that for $m > 0$,

$$\int_0^{\infty} \frac{\sin(mx)}{x} dx = \frac{\pi}{2}.$$

4) Prove that for every $\alpha > e$, $e^z = \alpha z$ has a solution in $D = \{z : |z| < 1\}$ while for $\alpha < \frac{1}{e}$, $e^z = \alpha z$ has no solutions in $D = \{z : |z| < 1\}$.

5) Find a conformal map from the quarter plane $\{z : Re(z) > 0, Im(z) > 0\}$ to the quarter circle $\{z : |z| < 1, Re(z) > 0, Im(z) > 0\}$ which takes $1+i$ to $\frac{1}{2}(1+i)$.

6)Suppose $f(z)$ is analytic in the complex plane $\mathbf{C}$ and satisfies

$$|f(z)| < C|z|^k$$

where C is independent of z and $k > 0$. Prove that $f(z)$ must be a polynomial.

7) Use complex variables to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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8) Suppose $\{f_n(z)\}$ is a family of analytic functions on a region G such that for each $z \in G$, there exists a neighborhood V with $z \in V \subset G$ and a constant C depending on V such that

$$|f_n(z)| < C$$

for all $z \in V$ and for all n. Prove that the family $\{f_n(z)\}$ is a normal family on G.

9) Prove that the only bounded harmonic functions in the entire complex plane are constants.

10) Prove that if $\{z_n\}$ is a sequence ofcomplex numbers tending to ∞ and $\{\alpha_n\}$ is an arbitrary sequence of complex numbers, there exists an entire function $f(z)$ such that

$$f(z_n) = \alpha_n.$$