COMPLEX ANALYSIS PRELIM

May, 1990

1. a) State the Riemann Mapping Theorem.

   b) Give an explicit conformal mapping which takes $C\setminus((-\infty,-1]\cup[1,\infty))$ onto the unit disc.

   c) Is the punctured disc $C\setminus\{0\}$ conformally equivalent to any annulus $\{ z \in C : a < |z| < b \}$ with a, b finite ? Justify your answer.

2.) Let Γ be the square with corners $-i$, $+i$, $-2+i$, $-2-i$, traversed counterclockwise; let $f(z) = 2/(z^3-1)$. Compute the winding number of the curve $f(\Gamma)$ about the point $w = -1$, giving $f(\Gamma)$ the orientation it inherits from Γ.

3.) Calculate $\int_0^\infty \frac{\cos(x)}{1 + x^2}\, dx$.

4.) Compute the Taylor expansion, about the origin, of $f(z) = \arctan(z^2)$, and determine its radius of convergence.

5.) The Gamma function is defined for $Re(z) > 0$ by

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt .$$

   a) Show that $\Gamma(z)$ extends to a meromorphic function on the whole complex plane.

   b) Determine the locations of the poles of $\Gamma(z)$, their orders, and their residues.

6.) Describe the Riemann surface of the algebraic function $f(z) = \sqrt{z^3-z}$, including branch points and the behavior at ∞.

7.) Construct a function analytic on the unit disc, which has a zero of order n at $z_n = 1 - 1/n$, for $n = 1,2,3,\ldots$.

8.) Let $f(z)$ be an entire function with the property that $|f(z)| \le |f(z^2)|$ for all z. Show that $f(z)$ must be constant.