QUALIFYING EXAMINATION IN COMPLEX ANALYSIS
August 12, 2011, 9–11 a.m.

\( \mathbb{D} \) denotes the (open) unit disk, \( \overline{\mathbb{D}} \) the closed unit disk, and \( \mathbb{T} = \partial \mathbb{D} \) the unit circle. Provide justifications as appropriate.

1. (15 points) Use methods of complex analysis to evaluate

\[
\int_{0}^{\infty} \frac{\sqrt{x}}{(x + 1)^2} \, dx
\]

Be sure to provide complete justifications.

2. (15 points) Let \( c \) be a complex number such that \( |c| < 1/3 \). Show that on the open set \( \text{Re}(z) < 1 \) the function \( f(z) = ce^z \) has exactly one fixed point, i.e., a point \( z_0 \) such that \( f(z_0) = z_0 \).

3. (20 points) Let \( \overline{B}(a, r) \) denote the closed disk of radius \( r > 0 \) about a point \( a \in \mathbb{C} \). Let \( f \) be a holomorphic function on an open set containing \( \overline{B}(a, r) \), and let \( M = \sup_{z \in \overline{B}(a,r)} |f(z)| \). Prove that for \( z \in \overline{B}(a, r/2) \), \( z \neq a \), we have

\[
\frac{|f(z) - f(a)|}{|z-a|} \leq \frac{2M}{r}.
\]

4. (15 points) Suppose \( \Omega \subset \mathbb{C} \) is a region containing \( \mathbb{D} \) and \( f \) is holomorphic on \( \Omega \). Suppose that on \( \mathbb{D} \) we have \( f(z) = \sum a_n z^n \) and the series has radius of convergence equal to 1.
   a. Give an example of such an \( f \) so that the series converges at every point of \( \mathbb{T} \).
   b. Give an example of such an \( f \) that is analytic at \( z_0 \in \mathbb{T} \) and for which \( \sum a_n z_0^n \) diverges.
   c. Prove that \( f \) cannot be analytic at every point of \( \mathbb{T} \).

5. (20 points) Consider \( f(x, y) = x^2 - 2y + y^3 \). Let \( P = (1, 1) \).
   a. Let \( X = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\} \). Prove that there is a neighborhood \( U \) of \( P \) so that \( U \cap X \) is given by \( y = \phi(x) \) for some \( C^1 \) function \( \phi \). Give \( \phi'(x) \).
   b. Now consider the same equation in \( \mathbb{C}^2 \) (i.e., \( Y = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\} \)). Prove that the analogous statement holds. What is \( \phi'(x) \) as an \( \mathbb{R} \)-linear map from \( \mathbb{C} \) to \( \mathbb{C} \)? Why is \( \phi \) holomorphic? (Hint: If you consider \( g : \mathbb{C} \to \mathbb{C} \) as a map \( \tilde{g} : \mathbb{R}^2 \to \mathbb{R}^2 \), how is holomorphy of \( g \) characterized by \( D \tilde{g} \)?)

6. (15 points) Do either a or b.
   a. Suppose \( f \) is meromorphic on \( \mathbb{D} \), continuous on \( \overline{\mathbb{D}} \) except at finitely many points of \( \mathbb{D} \), and real on \( \mathbb{T} \). Prove that \( f \) is a rational function.
   b. Let \( \Omega \subset \mathbb{C} \) be the region inside the unit circle \( |z| = 1 \) and outside the circle \( |z - \frac{1}{4}| = \frac{1}{4} \). Find a one-to-one conformal map from \( \Omega \) onto an annulus \( r < |z| < 1 \) for the appropriate value of \( r \).