F98

Complex Analysis Prelim

1. (a) Determine a linear fractional transformation

   $T : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \quad (\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\})$ such that

   $T(0) = 1, T(1) = -1$ and $T(\infty) = i$.

   (b) Determine $T(\hat{\mathbb{R}})$ and $T(\{z : Imz > 0\})$.

2. Let $\{f_n : n \geq 1\}$ be a sequence of holomorphic functions on $\Omega = \{z : |z| < 1\}$ converging uniformly on compact subsets of Ω to f with $f(0) = 0$. Prove either $f = 0$ or in every neighborhood of 0 infinitely many of the $f_n's$ have a zero.

3. Let f be an entire function. Suppose that for some $R > 0$ there is an $M > 0$ such that
$$|f(z)| \leq M|z|^n$$
for all $|z| \geq R$. Prove f is a polynomial of degree $\leq n$.

4. Prove there is an analytic function f defined on
$$\{z : |z| > 2\} \text{ such that } f(z)^2 = z^2 - 1 \text{ for all } |z| > 2.$$

5. Let f be analytic on $\Omega = \{z : |z| < 1\}$ and $\sup|f(z)| \leq 1$.

   Prove that for any $a \in \Omega$
$$\frac{|f'(a)|}{1 - |f(a)|^2} \leq \frac{1}{1 - |a|^2}.$$

6. Evaluate
$$\int_0^\infty \frac{1}{1 + t^4} dt.$$

7. Suppose C is a simple closed curve in the complex plane bounding the region D, with f and g holomorphic functions on the closure of D.

If $|f(z)-g(z)| < |f(z)|+|g(z)|$ for all $z \in C$ prove f and g have the same number of zeros in D.

8. Consider the function f where

$$f(z) = \frac{1}{(z-1)(z-2)}$$

   (a) Determine the Laurent expansion for $0 < |z-1| < 1$.

   (b) Determine the Laurent expansion for $1 < |z-1| < \infty$.

9. Suppose $a_n \geq 0$ for $n = 1, 2, \ldots$. Prove $\prod_{n=1}^{\infty} (1 - a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n$ converges.