Complex Analysis Qualifying Examination

Spring 2009

Show your work and carefully justify/prove your assertions.

All problems are of equal weight. Please arrange your solutions in numerical order even if you do not solve them in that order.

1. Evaluate $\int_0^\infty \frac{\sin x}{x}\, dx$ using complex analysis (other methods will not count).

2. Consider the transformation $w = z + \frac{1}{z}$ where $w = u + iv$ and $z = x + iy$. Let Ω be the domain consisting of all the points in the upper half plane $y > 0$ which are exterior to the circle $|z| = 1$.

   (a) Show that the image of Ω is the entire half plane $v > 0$.

   (b) What is the image of the boundary of Ω?

3. (a) Let $u(x, y) = y^3 - 3x^2 y$. Find its *harmonic conjugate* $v(x, y)$.

   (b) Let Ω be a connected domain in the complex plane and let u be harmonic in Ω. Prove that u satisfies the *mean value property* in Ω. That is, for each a in Ω there exists $r_0 > 0$ such that

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})\, dt \ \text{ for } \ 0 < r \leq r_0 \,.$$

4. How many roots does the equation

$$z^7 - 4z^3 + z - 1 = 0$$

   have in the open disk $|z| < 1$?

5. Describe the set $\mathrm{Aut}(\mathbb{C}_\infty)$ of analytic isomorphisms on $\mathbb{C}_\infty$ explicitly, where $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. Here a function f that has a pole at z_0 is considered to be analytic at z_0 as a mapping that sends z_0 to the point ∞ on the Riemann sphere $\mathbb{C}_\infty$.

6. (a) Show that $\frac{\pi^2}{\sin^2 \pi z}$ and $g(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ have the same principle part at each integer point.

   (b) Conclude that $\frac{\pi^2}{\sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

GOOD LUCK!