Complex Analysis Qualifying Exam, Fall 2010

Do all problems, and justify your assertions.

(1) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$.

(a) Define what it means for f to be differentiable at a point $(a, b) \in \mathbb{R}^2$ (in terms of linear transformations).

(b) State a version of the inverse function theorem in this setting.

(c) Identifying $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, give, with proof, a necessary and sufficient condition for a function satisfying the definition of *real* differentiability in part (a) to be *complex* differentiable at the point $a + bi$.

(2) Let $a > 0$. Evaluate $\int_0^\infty \frac{x^2}{(x^2+a^2)^3}\,dx$.

(3) Let f be entire. Discuss, with proofs and examples, the types of singularities f might have at ∞ in each of the following cases:

(a) f has at most finitely many zeros in $\mathbb{C}$;

(b) f has infinitely many zeros in $\mathbb{C}$.

(4) Let $\{f_n\}$ be a sequence of entire functions. Suppose $\{f_n\}$ converges pointwise to a function $g : \mathbb{C} \to \mathbb{C}$, and the convergence is uniform on each line segment in $\mathbb{C}$. Show that g is entire, and that $f_n \to g$ uniformly on each compact subset of $\mathbb{C}$.

(5) Let $H = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Supppose f is an analytic function which takes the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$ to H, and satisfies $f(0) = 2$. Find a sharp upper bound for $|f'(0)|$, justifying your bound by a proof and its sharpness by an example.

(6) Let u, v be harmonic functions on a region G. Prove that if the product uv is identically zero, then either u or v must be identically zero.