# Complex Analysis Qualifying Examination

**Fall 2008**

Show work and carefully justify/prove your assertions.

Notation: $\mathbb{D}, \mathbb{H}$ denote respectively the unit disc centered at origin and the upper half plane.

*All problems are of equal weight. Do the easier ones first. Please arrange your solutions in numerical order even if you do not solve them in that order.*

1. Evaluate $\displaystyle\int_0^\infty \frac{1-\cos ax}{x^2}\,dx$ using complex analysis, where $a \neq 0$ is real.

2. Find a conformal map from $\{z : |z-i| > 1, \operatorname{Im}(z) > 0\}$ to $\mathbb{H}$.

3. Suppose $f$ is a non-vanishing continuous function on $\overline{\mathbb{D}}$ that is holomorphic in $\mathbb{D}$. Prove that if
$$|f(z)| = 1 \quad \text{whenever } |z| = 1,$$
then $f$ is a constant.

4. Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large $n$'s.

5. Let $z_0$ be a pole for $f(z)$. Show that $z_0$ is an essential singularity for $\sin(f(z))$.

6. Assume $f(z)$ is analytic in $\mathbb{D} : |z| < 1$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta}z$). Show that $\displaystyle\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$, where $f^1 = f$ and $f^{n+1}(z) = f(f^n(z))$.

$f : \mathbb{D} \to \mathbb{D}$ is assumed