Complex Analysis Qualifying Examination — Fall 2006

Show work and carefully justify/prove your assertions.

1. Evaluate $\int_0^\infty \frac{x^2}{1+x^4}\,dx$ using the method of complex contour integration. Justify all steps.

2. Find a conformal map from the intersection of $|z-1|<2$ and $|z+1|<2$ to the upper half plane. (Note that the intersection of the circles $|z-1|=2$ and $|z+1|=2$ are at $z=\pm i\sqrt{3}$.)

3. Explicitly describe all entire functions f such that $|f(z)| \geq |z|$ for all $z \in \mathbb{C}$.

4. (1) State the Riemann mapping theorem.

   (2) Let U be a simply-connected open and proper subset of $\mathbb{C}$, and let $f: U \to U$ be a holomorphic bijection. Prove or disprove (i.e. give a counter-example to) the following assertion: if f has two fixed points, then $f(z) = z$ for all $z \in U$.

5. Let $f(z)$ be analytic in a domain D. Fix $z_0 \in D$ and let $w_0 = f(z_0)$. Suppose z_0 is a zero of finite order m (i.e. multiplicity m) for $f(z) - w_0 = 0$. Show that there exist $\delta > 0$ and $\bar{\delta} > 0$ such that for each w with $0 < |w - w_0| < \bar{\delta}$, the equation $f(z) - w = 0$ has exactly m *distinct* solutions inside the disk $|z - z_0| < \delta$.