Complete all 6 problems. In multi-part problems, you may assume the result of any part (even if you have not been able to do it) in working on subsequent parts. Please fully justify all your answers. (Points are distributed evenly between parts.)

1. (Group Actions, 10 points) Let G be a finite group, and let X denote a finite set with a G-action. Let X^G denote the set of fixed points of the action, so $X^G = \{x \in X | gx = x \text{ for all } g \in G\}$.

   (a) Suppose G is a finite p-group. Prove that $|X^G| \equiv |X| \mod p$.

   (b) Suppose G is a finite p-group. Assume in addition that $p \not| |X|$. Prove that this action has a fixed point.

2. (Sylow Theorems, 10 points) Let G be a finite group, P a p-Sylow subgroup of G, and denote by $N_G(P)$ the G-normalizer of P in G.

   (a) Prove that P is the unique p-Sylow subgroup of $N_G(P)$.

   (b) How many elements of order 5 does a non-cyclic group of order 55 have?

3. (Rings, 15 points) Let F be a field, and consider the following subset of polynomials

$$R = \{f(x) \in F[X] | \text{ the } x\text{-coefficient of } f(x) \text{ is } 0_F\}.$$

   (a) Prove that R is a subring of $F[x]$.

   (b) Prove that R is not a UFD.

   (c) Prove that R is not a PID.

4. (Field extensions and Galois Theory, 15 points) Suppose $K \subseteq \mathbb{C}$ is a splitting field for $\mathbb{Q}$ for $x^3 - 2$.

   (a) Compute $[K : \mathbb{Q}]$, and give a $\mathbb{Q}$-basis for K.

   (b) What is the Galois group $G = G(K : \mathbb{Q})$?

   (c) Find all subfields of K.

5. (Modules, 15 points) Fix a ring R. A sequence of R-modules and homomorphisms between them $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ is called *exact* at M_2 if $\mathrm{Im} f_1 = \ker f_2$.

   (a) Prove that a sequence $0 \to A \xrightarrow{\alpha} B$ is exact at A if and only if α is a monomorphism.

   (b) Prove that a sequence $B \xrightarrow{\beta} C \to 0$ is exact at C if and only if β is an epimorphism.

   (c) Suppose the sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact at A, B, and C. Assume in addition that there is a homomorphism $\tau : B \to A$ so that $\tau \circ \alpha = \mathrm{id}_A$. Give an explicit isomorphism $\phi : B \to A \oplus C$, and show it is an isomorphism.

6. (Jordan Canonical Form, 15 points) Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \in M_3(\mathbb{C}).$$

   (a) Find the Jordan canonical form J of A.

   (b) Fix a complex number c. Suppose V is a complex vector space of dimension 5, and let T be a linear operator on V whose characteristic polynomial is $(t - c)^5$. Suppose that the rank of $T - cI$ is 2. What are the possible Jordan forms for T?

   (c) What are the possible Jordan forms for a matrix whose characteristic polynomial is $(t-2)^2(t-5)^3$ when the space of eigenvectors with eigenvalue 2 is 1-dimensional and the space of eigenvectors with eigenvalue 5 is 2-dimensional?