ALGEBRA QUALIFYING EXAM, JANUARY 2017

Problems 2 and 8 are worth 15 points each, and all others are worth 10 points (90 points total).

(1) Let G be a finite group and let $\pi : G \to \mathrm{Sym}(G)$ be the Cayley representation. (Recall that this means that for an element x in G , $\pi(x)$ acts by left translation on G.) Prove that $\pi(x)$ is an odd permutation if and only if the order $|\pi(x)|$ of $\pi(x)$ is even and $|G|/|\pi(x)|$ is odd.

(2) (a) How many isomorphism classes of abelian groups of order 56 are there? Give a representative for one of each class.

(b) Prove that if G is a group of order 56, then either the Sylow-2 subgroup or the Sylow-7 subgroup is normal.

(c) Give two non-isomorphic groups of order 56 where the Sylow-7 is normal, and the Sylow-2 subgroup is not normal. Justify that your two groups are not isomorphic.

(3) Let R be a commutative ring with 1. Suppose that M is a free R-module with a finite basis X.

(a) Let I be a proper ideal of R. Prove that M/IM is a free R/I-module with basis X', where X' is the image of X under the canonical map $M \to M/IM$.

(b) Prove that any two bases of M have the same number of elements. You may assume that the result is true when R is a field.

(4) (a) Let R be an integral domain with quotient field F. Suppose that $p(x), a(x)$, and $b(x)$, are monic polynomials in $F[x]$ with $p(x) = a(x)b(x)$, and with $p(x)$ in $R[x]$, $a(x)$ not in $R[x]$, and both $a(x)$ and $b(x)$ not constant. Prove that R is not a UFD. You may assume Gauss' lemma.

(b) Prove that $\mathbb{Z}[2\sqrt{2}]$ is not a UFD. Hint: let $p(x) = x^2 - 2$.

(5) Let R be an integral domain, and let M be a nonzero torsion R-module.

(a) Prove that if M is finitely generated then the annihilator in R of M is nonzero.

(b) Give an example of a non-finitely generated torsion R-module whose annihilator is (0). Justify your answer.

(6) Let A be an $n \times n$ matrix with all entries equal to 0 except for the $n-1$ entries just above the diagonal being equal to 2.

(a) What is the Jordan Canonical Form of A, viewed as a matrix in $M_n(\mathbb{C})$?

(b) Find a non-zero matrix $P \in M_n(\mathbb{C})$ such that $P^{-1}AP$ is in Jordan Canonical Form.

(7) Let F be a field and let $f(x) \in F[x]$.

(a) Define what is a splitting field of $f(x)$ over F.

(b) Let F now be a finite field with q elements. Let E/F be a finite extension of degree $n > 0$. Exhibit an explicit polynomial $g(x) \in F[x]$ such that E/F is a splitting field of $g(x)$ over F. Fully justify your answer.

(c) Show that the extension E/F in (b) is a Galois extension.

(8) (a) Let K denote the splitting field of $x^5 - 2$ over $\mathbb{Q}$. Show that the Galois group of $K/\mathbb{Q}$ is isomorphic to the group of invertible matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ where $a \in \mathbb{F}_5^*$ and $b \in \mathbb{F}_5$.

(b) Determine all intermediate fields between the splitting field K of $x^5 - 2$ and $\mathbb{Q}$ which are Galois over $\mathbb{Q}$.