ALGEBRA QUALIFYING EXAM, SPRING 2026

Instructions: Complete all 7 problems. Each problem is worth 10 points. In multi-part problems, you may assume the result of any part (even if you have not been able to do it) in working on subsequent parts.

(1) (a) State the three Sylow theorems.
(b) Prove the third Sylow theorem (the one about the number of Sylow subgroups). You may use the first two Sylow theorems freely.
(c) Show that there are no simple groups of order 105.

(2) (a) Suppose a finite group G acts transitively on a set X, and $|X| \geq 2$. Show that there is a $g \in G$ that does not fix any points in X.
(b) Let G be finite group of even order. Show that there are an odd number of elements of order 2.
(c) Let G be a finite p-group, and let $H \subseteq G$ be a non-trivial normal subgroup. Show that $H \cap Z(G) \neq \{e\}$.

(3) (a) Define the Galois group of an irreducible polynomial $p(x) \in \mathbb{Q}[x]$.
(b) Show that $p(x) = x^3 + 3x^2 + 3x + 4$ is irreducible in $\mathbb{Q}[x]$.
(c) Compute the Galois group of $p(x)$.

(4) Let $\mathbb{C}[[x]]$ denote the ring of complex formal power series, $f(x) = a_0 + a_1x + a_2x^2 + \ldots$ with each $a_i \in \mathbb{C}$.
(a) Show that $\mathbb{C}[[x]]$ forms an integral domain.
(b) Compute the units in $\mathbb{C}[[x]]$.
(c) Show that every ideal $\mathbb{C}[[x]]$ is of the form $\langle x^n \rangle$.
(d) Up to isomorphism, how many $\mathbb{C}[[x]]$-modules are there that are 6-dimensional as vector spaces over $\mathbb{C}$?

(5) Consider the following 3×3-matrix.

$$B = \begin{pmatrix} 8 & -8 & -1 \\ 1 & -1 & -1 \\ 9 & -8 & -2 \end{pmatrix}$$

(a) Find the characteristic and minimal polynomials of B.
(b) Find a 3×3 matrix J in Jordan canonical form such that $B = PJP^{-1}$ where P is an invertible matrix.

(6) (a) True or False: $\mathbb{F}_{25}$ is isomorphic to a subfield of $\mathbb{F}_{125}$. Justify your answer
(b) Let $f(x)$ be an irreducible factor of $x^{p^k} - x$ over $\mathbb{F}_p$. Show that the degree of $f(x)$ divides k.
(c) How many irreducible factors of degree $1, 2, 3$, and 6 are there when you factor $x^{5^6} - x$ over $\mathbb{F}_5$? You do not need to list the factors.

(7) Let R be a commutative ring, and let M be an R-module. An R-submodule N of M is maximal if there is no R-module P with $N \subsetneq P \subsetneq M$.
   (a) Show that an R-submodule N of M is maximal iff M/N is a simple R-module (that is, M/N is nonzero and has no proper, nonzero R-submodules.)
   (b) Let M be a $\mathbb{Z}$-module. Show that a $\mathbb{Z}$-submodule N of M is maximal iff $|M/N|$ is a prime number.
   (c) Let M be the $\mathbb{Z}$-module of all roots of unity in $\mathbb{C}$ under multiplication. Show that there is no maximal $\mathbb{Z}$-submodule of M.