ALGEBRA QUALIFYING EXAM FALL 2024

(1) (12 points) Let G be a group and H, K subgroups of G. Recall that HK is defined as $HK = \{hk | h \in H, k \in K\}$.

(a)If K is normal in G prove that HK is a subgroup of G. (3 points)

(b)Give an example of a group G with subgroups H and K where HK is not normal (3 points)

(c)Suppose that G is finite, K is normal in G and P is a Sylow-p subgroup of K. Let $g \in G$. Prove that there is an element $k \in K$ such that $gk^{-1} \in N = N_G(P)$. (Recall that $N_G(P)$ is the normalizer of P in G.) (5 points)

(d)Let G, K, N be as in part (c). Prove that $G = NK$. (1 point)

(2) (12 points)(a)State and prove the Second Isomorphism Theorem for groups.You may assume the First Isomorphism Theorem. (Hint:The First Isomorphism Theorem relates the image of a homomorphism to a quotient group . The Second Isomorphism Theorem applies to a situation such as part (a) of the previous problem. It is sometimes called the Diamond Isomorphism Theorem because of the diamond shaped lattice of subgroups of G involved.) (6 points)

(b)Let $n \geq 5$. Use the Second Isomorphism Theorem for groups and the fact that A_n is simple to prove that the only nontrivial normal subgroup of S_n is A_n.(6 points)

(3) (12 points) Let R be a Principal Ideal Domain.

(a)Let $I_1 \subseteq I_2 \subseteq \cdots \subseteq R$ be an ascending chain of ideals in R. Prove that for some positive integer n, $I_k = I_n$ for all $k \geq n$ (6 points) .

(b)Prove that every element in a R has a factorization into irreducibles. (6 points)

(4) (12 points) Let R be a commutative ring with 1. If I is an ideal in R and M is an R-module define IM to be the collection of elements consisting of all finite sums of the form $\sum a_i m_i$ where $a_i \in I$ and $m_i \in M$.

(a)Prove that IM is a submodule of M. (3 points)

(b)If I, J are ideals in R, define a map $\phi : M \to M/IM \times M/JM$ by $x \to (x + IM, x + JM)$. Prove that this map is a R-module homomorphism with kernel $IM \cap JM$. (3 points)

(c)With the notation in (b), assume also that $I + J = R$. Prove that $M/(IJ)M \cong M/IM \times M/JM$.(6 points)

(5) (6 points) Let F be a field and let $f(x) \in F[x]$. Assume that F contains all the roots of $f(x)$. Prove that all matrices with characteristic polynomial $f(x)$ are similar if and only if $f(x)$ has no repeated factors in its unique factorization in $F[x]$.

(6) (10 points) Let F be a field.

(a)Use the fact that the polynomial ring $F[x]$ has a Euclidean division algorithm to prove that every ideal in $F[x]$ is principal. (5 points)

(b)Let E be an extension field of F and let $\alpha \in E$ be an element of E which is algebraic over F. Let $F[\alpha] = \{f(\alpha) | f(x) \in F[x]\}$. Assume that the evaluation map $\phi : F[x] \to F[\alpha]$ which maps $f(x) \to f(\alpha)$ is a homomorphism. Prove that $F[\alpha]$ is a field which is contained in every field containing α. You may assume that if $f(x)$ is irreducible in $F[x]$ that $F[x]/(f(x))$ is a field.(5 points)

(7) (16 points)In the following problem, for an extension field E of F,we will use the notation $Aut(E/F)$ to denote the group of automorphisms of E which fix the elements of F (the Galois group of E over F).

Let $f(x) = x^4 - 2 \in \mathbb{Q}[x]$ and let $K = \mathbb{Q}(2^{1/4}, i)$.

(a)Prove that K is a splitting field for $f(x)$. (1 point)

(b)Let G be the Galois group for $f(x)$ over Q. Prove that there is an element of G, call it σ , which maps $2^{1/4} \to 2^{1/4}i, i \to i$, and an element of G , call it τ , which maps $2^{1/4} \to 2^{1/4}, i \to -i$. (5 points)

(c)Identify the group G in (b). (4 points)

(d)(6 points)For the following intermediate fields, $\mathbb{Q} \subseteq E_i \subseteq K$, find their Galois groups $Aut(K/E_i)$ as subgroups of G. Determine which are Galois over $\mathbb{Q}$. For those which are Galois over $\mathbb{Q}$, identify $Aut(E_i/\mathbb{Q})$ as a quotient group of G. Justify your answers.

(i)$E_1 = \mathbb{Q}(2^{1/4})$.

(ii)$E_2 = \mathbb{Q}(i)$.

(iii)$E_3 = \mathbb{Q}(\sqrt{2}, i)$.

(iv)$E_4 = \mathbb{Q}(\sqrt{2})$.

(v)$E_5 = \mathbb{Q}(2^{1/4}i)$.